Frames for Mathematical Proofs

Workshop “Mathematical Language and Practical Type Theory”

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Outline

Motivation

Assumption: Different layers of interpretation of a mathematical text are useful at different stages of analysis and in different contexts.

Immediate Goal: make explicit in the formal representation of information that is implicit in the textual form

Theoretical Goal: bridge gap between formalist and textualist positions regarding mathematical proofs

Tools: from formal linguistics and artificial intelligence

Theses

1. FRAMES can serve as the basis for describing mathematical proofs.
2. Specifically, using FRAMES it is possible to model how mathematicians understand proofs that conform to proof patterns which have not been executed in a fully explicit way.
3. FRAMES can be used to model both (textual) structural properties of proofs and ontological aspects of mathematical knowledge. This distinction is useful.

Outline

What are Frames?

Properties

- a concept in knowledge representation
  ⇝ Fillmore (1968) and Minsky (1974)
- represent conceptual structure or prototypical situations
  e.g. ***birthday celebration***, ***restaurant***.
- ***roles*** and ***participants*** (slots and fillers)
  e.g. *waiter*, *diners*, *food*, ...
- organized in an ***inheritance hierarchy***
  typed *feature structures* (Carpenter, 1992)

Usage

- e.g., in cognitive linguistics and artificial intelligence
- explain how receiver completes information conveyed by sender

⇝ linguistic project: FrameNet database (1,200 semantic frames)

Framing and Frames

One event can be framed differently, e.g. as ***buying*** and as ***selling***

Frame: BUYING

[BUYER John] **bought** [GOODS a beautiful medieval book] [TIME yesterday].

Frame: SELLING

[SELLER Petra] **sold** [GOODS a beautiful medieval book] to [BUYER John] for [MONEY twenty Euros].

Frames as feature structures

$$
\begin{bmatrix}
\textit{buy} & \\
\textsc{Buyer!} & [\![\textit{John}]\!] \\
\textsc{Goods!} & [\![\textit{a beautiful medieval book}]\!] \\
\textsc{Time} & [\![\textit{yesterday}]\!] \\
\textsc{Seller} & \textit{person} \\
\textsc{Money} & \textit{money} \\
\textsc{Purpose} & \textit{purpose} \\
\dots &
\end{bmatrix}
=
\begin{bmatrix}
\textit{buy} & \\
\textsc{Buyer!} & j \\
\textsc{Goods!} & b \\
\textsc{Time} & \begin{bmatrix}
\textit{point-in-time} & \\
\text{YEAR} & 2019 \\
\text{MONTH} & 08 \\
\text{DAY} & 06 \\
\text{HOUR} & \{1,\dots,24\} \\
\text{MINUTE} & \{0,\dots,60\} \\
\dots &
\end{bmatrix} \\
\textsc{Seller} & \textit{person} \\
\textsc{Money} & \textit{money}
\end{bmatrix}
$$

The Commercial Transaction frame from FrameNet

Frame Index

Commercial_transaction

Lexical Unit Index

Definition:

These are words that describe basic commercial transactions involving a Buyer and a Seller who exchange Money and Goods. The individual words vary in the frame element realization patterns. For example, the typical patterns for the verbs buy and sell are: BUYER buys GOODS from the SELLER for MONEY. SELLER sells GOODS to the BUYER for MONEY.

His $20 TRANSACTION with Amazon.com for a new TV had been very smooth.

FEs:

Core:

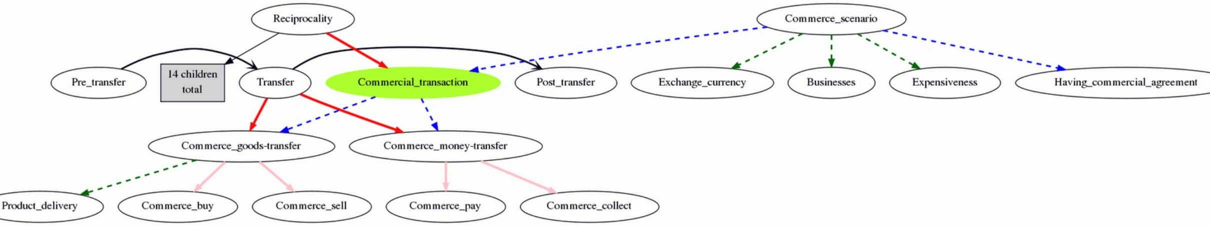
Buyer [Byr]	The Buyer wants the Goods and offers Money to a Seller in exchange for them.
Goods [Gds]	The FE Goods is anything (including labor or time, for example) which is exchanged for Money in a transaction.
Money [Mny]	Money is the thing given in exchange for Goods in a transaction.
Seller [Slr]	The Seller has possession of the Goods and exchanges them for Money from a Buyer.

Non-Core:

Means [Mns] **Semantic Type:** State_of_affairs	The means by which a commercial transaction occurs.
Rate [Rate]	Price or payment per unit of Goods.
Unit [Unit]	The Unit of measure of the Goods according to which the exchange value of the Goods (or services) is set. Generally, it occurs in a by-PP.

Frame-frame Relations:

Frame-to-Frame relations: Multiple Inheritance



Screenshot https://framenet.icsi.berkeley.edu/fndrupal/FrameGrapher

Outline

Frames in Mathematical Texts

Goal: Model proofs and proof methods

Types of frames: (define types of slots)

Ontological: type of mathematical object

e.g. **Circle**, **slots:** center, radius, diameter, circumference, ...

e.g. **Vector Space**, **slots:** zero, unit, field, dimension, ...

Structural: part of proofs

e.g. **Induction**, **slots:** induction variable, hypothesis, step, domain, ...

e.g. **Extremal Proof**, **slots:** object type, initial object, parameter

Frames Example: Induction

Induction Frame (structural)

with slots:

- Base Case
- Induction Hypothesis
- Induction Step
- Induction Domain: **Inductive Type** (**ontological**) with
  - Base Constructor
  - Recursive Constructor
- Induction variable

(see Fisseni, Sarikaya, Schmitt and Schröder, 2019) *induction*

- INDUCTION-DOMAIN [d]
  - *inductive-type*
  - BASE-CONSTRUCTOR [bc] *base-constructor*
  - RECURSIVE-CONSTRUCTOR [rc] *recursive-constructor*
- INDUCTION-VARIABLE
  - *variable*
  - NAME [x] *symbolic*
  - TYPE [d]
- ASSERTION ∀[x]. [s]
- PROOF
  - *induction-proof*
  - INDUCTION-SIGNATURE
    - *induction-signature*
    - INDUCTION-HYPOTHESIS [ih] *sentence*
    - STEP-FUNCTION (?!) [rc]
    - BASE-CONDITION [bcc] (?!) [x] = [bc]
    - INDUCTION-CONDITION [icc] (?!) [x] = [rc](...)
  - BASE-CASE
    - *proved-under-hypothesis*
    - HYPOTHESIS [bcc]
    - THESIS [s]
    - ASSERTION [bcc] ⇒ [s]
    - PROOF *list(proof-step* ∨ *assumption* ∨ *definition* ∨ *goal)*

[...]

ASSERTION: ∀[x]. [s]

PROOF:
- *induction-proof*
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An Induction Proof

Kowalski (2016, p. 93)

Proof. First, the second statement is indeed more precise than the first: let $k \geq 1$ be such that $f^k = 0$ but $f^{k-1} \neq 0$; there exists $v \neq 0$ such that $f^{k-1}(v) \neq 0$, and we obtain $k \leq n$ by applying the second result to this vector v. We now prove the second claim. Assume therefore that $v \neq 0$ and that $f^k(v) = 0$ but $f^{k-1}(v) \neq 0$. Let $t_0, \dots, t_{k-1}$ be elements of K such that

$$t_1 v + \cdots + t_{k-1} f^{k-1[sic!]}(v) = 0.$$

Apply f^{k-1} to this relation; since $f^k(v) = \dots = f^{2k-2[sic!]}(v) = 0$, we get

$$t_1 f^{k-1}(v) = t_1 f^{k-1}(v) + t_2 f^k(v) + \cdots + t_{k-1} f^{2k-2[sic!]}(v) = 0,$$

and therefore $t_1 f^{k-1}(v) = 0$. Since $f^{k-1}(v)$ was assumed to be non-zero, it follows that $t_1 = 0$.
Now repeating this argument, but applying f^{k-2} to the linear relation (and using the fact that $t_1 = 0$), we get $t_2 = 0$.
Then similarly we derive ***by induction*** that $t_i = 0$ for all i, proving the linear independence stated.

in the first equation, the exponent $k - 1$ has to be replaced by $k - 2$; in the line below and the second equation, $2k - 2$ by $2k - 3$.

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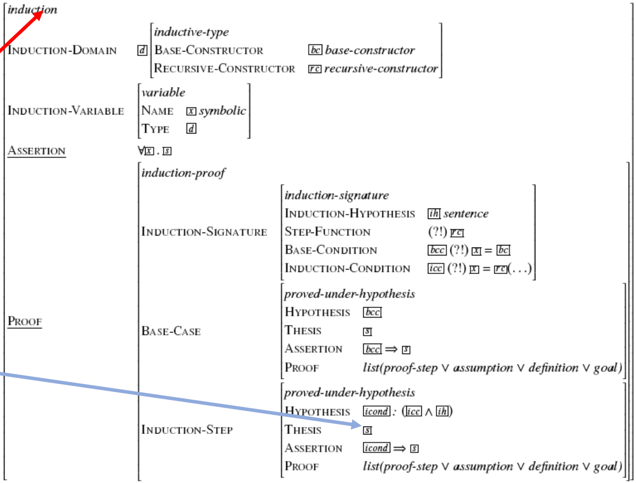

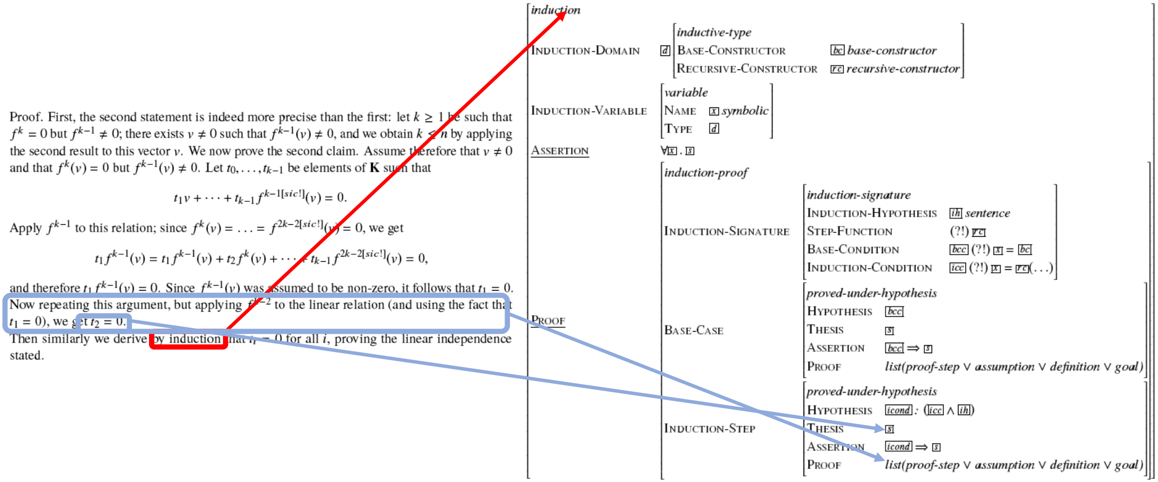
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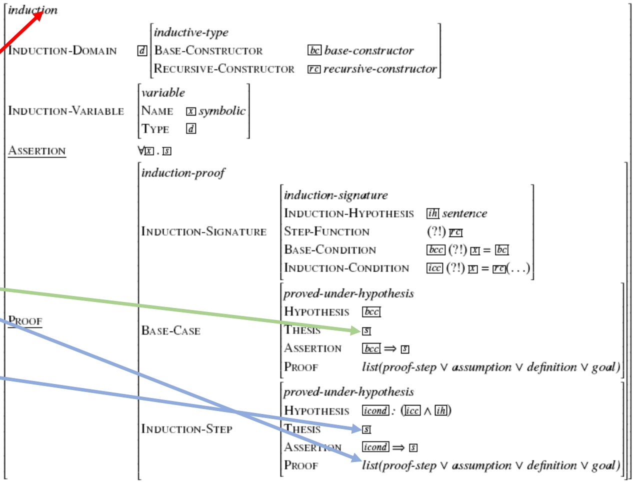

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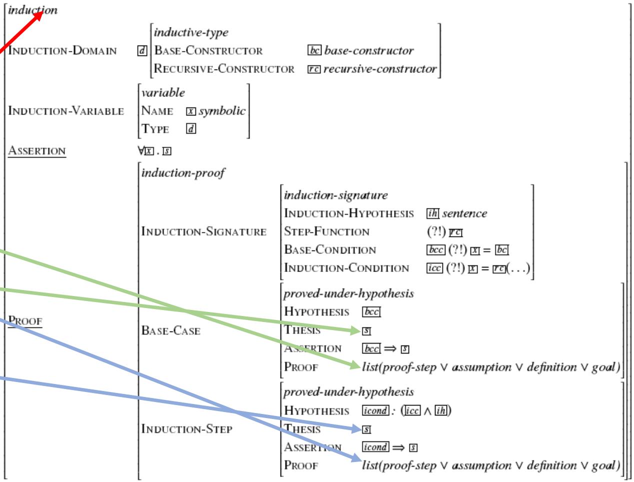
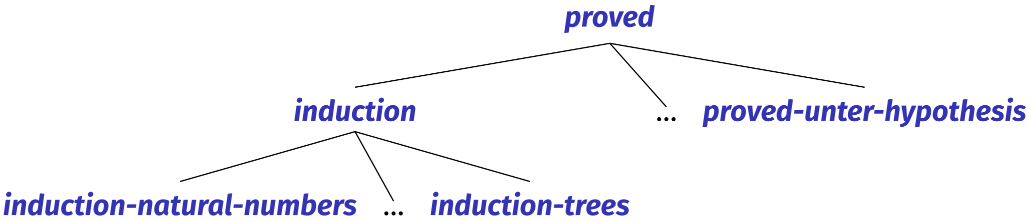


Inheritance hierarchy of proof frames



Outline

Further Frames

Kinds of frames and sources of information

- **Structural: Proof Techniques**, e.g. ENGEL's, 1999
- **Ontological: Domains**, e.g. MMT theories (RABE, 2016)

Another proof frame: *extremal* proof.

*"We are trying to prove the existence of an object with certain properties. The **extremal principle** tells us to pick an object which **maximizes** or **minimizes** some function. The **resulting object** is then shown to have the desired property by showing a slight perturbation (variation) would further increase or decrease the given function. [...]*

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(ENGEL, 1999, ***Problem-Solving Strategies***, p. 39)

Context and extremal proofs – interaction / ontological frames

„Das Extremalprinzip setzt also einen Kontext voraus, in dem minimale oder maximale Objekte existieren." (Carl, 2017, p. 75)

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Variations of extremal proofs

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Variations of extremal proofs

Carl: variation triggered by Engel's "three well-known facts", e.g.

domain natural numbers: triggers *least number principle*

domain subset of reals: triggers *least upper bound principle* or *largest lower bound principle*

Context and extremal proofs – interaction by hypothesis/goal

Carl (2017, p. 75): prototypical extremal arguments are different depending on the hypothesis:

Context and extremal proofs – interaction by hypothesis/goal

CARL (2017, p. 75): prototypical extremal arguments are different depending on the hypothesis:

> *Beweise mithilfe des Extremalprinzips funktionieren meist auf eine der beiden folgenden Weisen (two ways):*
>
> 1. *Zu zeigen ist eine **Existenzaussage** (existence statement). Das extremale Objekt ist ein **Beispiel** (example) für ein Objekt der gesuchten Art oder hilft bei dessen **Konstruktion** (construction).*
> 2. *Zu zeigen ist eine **Allaussage** (universal statement). Man nimmt das **Gegenteil** (opposite) an, betrachtet ein extremales **Gegenbeispiel** (counterexample) und arbeitet auf einen **Widerspruch** (contradiction) (meist zur Maximalität oder Minimalität) hin.*
>
> *CARL (2017, S. 75)*

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Frames and Mathematical Understanding

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$\leadsto$ related phenomenon: ***grasping*** a proof often linked to some figure of speech of ***zooming out***,
$\leadsto$ understanding needs knowing which frames were actually involved.

creativity: Poincaré saw creativity as (some fruitful) combination of old ideas or as choice among the manifold of all possible combinations.

proof identity: despite different surface stucture

Advantages of frame approach to mathematical texts

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Frames can serve as the basis for describing mathematical proofs:

- Frames may be cognitively real.
- Frames offer a new way to model gaps in proofs.
- Remark: Tactics as used in proof assistants can be modeled as frames.

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1. Frames can serve as the basis for describing mathematical proofs.
2. Specifically, using frames it is possible to model how mathematicians understand proofs that conform to proof patterns which have not been executed in a fully explicit way.
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Outlook

- more frames and more texts
- corpus-based annotation workflow
- didactic experiments

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