Sets and Games

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What is a Set?

Unter einer Menge verstehen wir jede Zusammenfassung M von bestimmten, wohlunterschiedenen Objekten m unser Anschauung oder unseres Denkens (welche die “Elemente” von M genannt werden) zu einem Ganzen.

Georg Cantor

“A set M is understood to be a collection of certain distinguishable objects m of our perception or our thoughts (which are called “elements” of M).”

Notation:

Let M and S be sets.

- $m \in M$ means m is an element of M.
- $\{a \,|\, P\}$ is the set of all a such that property P holds for a.
- $M \cup S$ is the union of the sets M and S.
- $M \cap S$ is the intersection of the sets M and S.
- $M \backslash S$ is the set we get when we take away all elements in M that are also in S.
- $M \subseteq S$ means that M is a subset of S.
- and finally, V denotes the universe of all sets.

Russell's Paradox :

Let X be the set of all sets that do not contain themselves.

$$X = \{M \mid M \notin M\}$$

Then, if X contains itself, it does not contain itself. ...

On the other hand, if X does not contains itself, it must contain itself, so either way we have a contradiction.

Some other paradoxes using self reference:

Epidemes Paradox:

"All cretians are always liars."

Barber Paradox:

In a small town lives a male barber who shaves daily every man that does not shave himself, and no one else.

A condensed Liar Paradox:

"This statement is not true"

The axioms of Set Theory:

- Existenz: There exists a set that contains no sets, the empty set.
- Extensionality: Two sets are the same if they contain exactly the same elements.
- Pairs: For every two sets exists a set that contains exactly those two sets.
- Union: For every set X exists a set $\bigcup X$ that contains exactly the elements of the elements of X.
- Power Set: For every set X exists the power set $\mathcal{P}(X)$, i.e. the set that contains all subsets of X.
- Infinity: There exists an infinite set.
- Separation: For every set X and every formula φ exists a set that contains exactly the elements of X for which φ holds.
- Replacement: For every set X and every functional formula φ exists a set Y, s.t. "Y is the image of X under φ."
- Foundation: If a formula φ is true for at least one set, then exists a set X s.t. φ ist true for X but not for any element of X.

Functions as sets:

Notation: $(a,b) := \{\{a\},\{a,b\}\}$, $A \times B := \{(a,b) \,|\, a \in A \text{ and } b \in B\}$.

Let f be a subset of $A \times B$, such that

- for all $a \in A$ there is a $b \in B$ s.t. $(a,b) \in f$,
- and if $(a,b) \in f$ and $(c,b) \in f$, then $a = c$

Then we call f a function from A to B.

Now properties of functions like surjectivity, domain, function value, etc. are just properties of the corresponding set.

Numbers as sets:

Let $0 := \{\}$ and $a + 1 := a \cup \{a\}$. Then all natural numbers can be viewed as sets:

$1 = 0 \cup \{0\} = \{\} \cup \{\{\}\} = \{\{\}\}$, $2 = 1 \cup \{1\} = \{0, 1\}$, $3 = \{0, 1, 2\}$, etc.

Now we can go on and formalize the integers as sets, the rational numbers, the reals, functions on reals, integrals, derivations ...

So all mathematics take place in the universe of sets, and set theory gives it a formal foundation.

Different Infinities:

There are countably infinitive many natural numbers. The size of the power set $\mathcal{P}(\mathbb{N})$ of the natural numbers is also infinite, but larger than countable infinite:

Assume we have a bijection $f\colon \mathbb{N} \to \mathcal{P}(\mathbb{N})$.

Let $X := \{z \mid z \in \mathbb{N}$ and $z \notin f(z)\}$.

Then $X \in \mathcal{P}(\mathbb{N})$, so there is an Y s.t. $X = f(Y)$.

If $Y \in f(Y)$, then by definition of X we have $Y \notin X = f(Y)$,

if $Y \notin f(Y)$, then by definition of X we have $Y \in X = f(Y)$,

either way we have a contradiction, so such a bijection can not exist.

The Axiom of Choice:

Take a finite collection of nonempty set. Easily one can choose an element from each set: Take an element from the first set in the collection, the one from the second, and so on.

Take an infinite collection of nonempty sets. Now this procedure will not work anymore, at least it will not come to an end in finite time.

The Axiom of Choice states that one can always choose, regardless of the size of the collection:

(AC) For every set $C = \{M_i \,|\, i \in I\}$ of nonempty sets exists a function $f\colon C \to \bigcup C$ with $f(M_i) \in M_i$.

Such a function is called a choice function.

Wellorderings:

A wellordering of a set M is a binary relation $\prec$ on M, s.t.

- For all $x \in M$ holds $x \not\prec x$.
- For all $x, y \in M$ either $x \prec y$ or $y \prec x$ or $x = y$.
- If $x \prec y$ and $y \prec z$ then $x \prec z$.
- Every subset of M has a $\prec$-minimal element.

So there are no infinitively descending chains in a wellordering.

Wellorderings of numbers:

The natural numbers are wellordered by $<$,

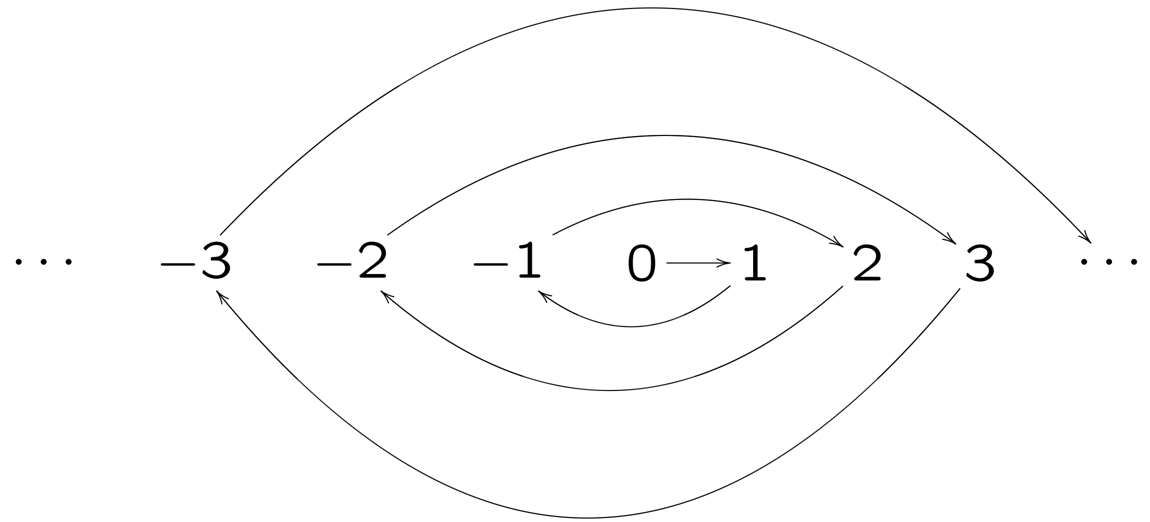
$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow 6 \longrightarrow 7 \longrightarrow 8 \longrightarrow 9 \longrightarrow 10 \longrightarrow \cdots$$

whereas the integers are not wellordered by $<$:

$$\cdots \longrightarrow -5 \longrightarrow -4 \longrightarrow -3 \longrightarrow -2 \longrightarrow -1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow \cdots$$

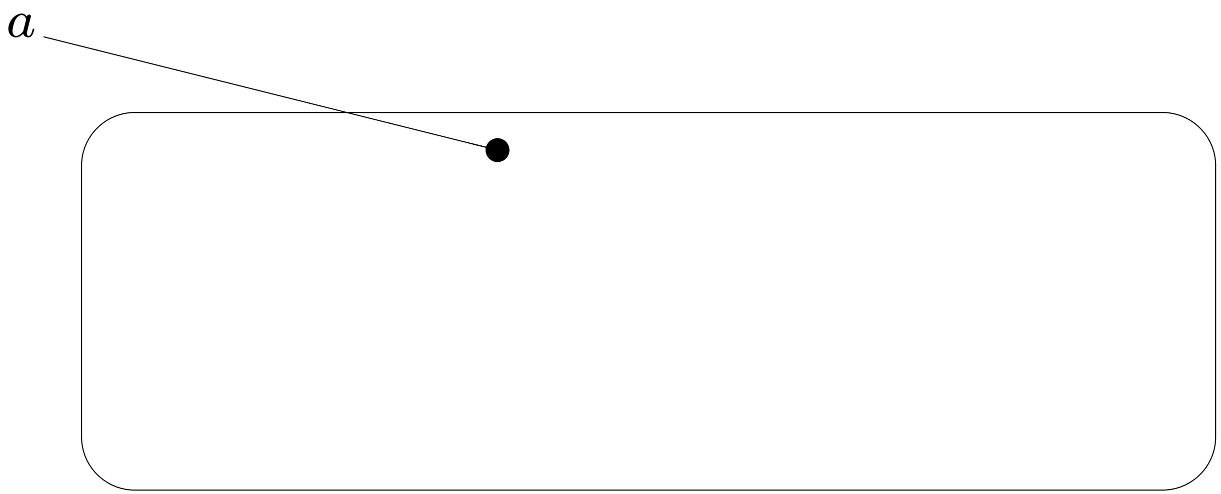
But we can wellorder the integers, using another order $\prec$:

$$0 \prec 1 \prec -1 \prec 2 \prec -2 \prec 3 \cdots$$



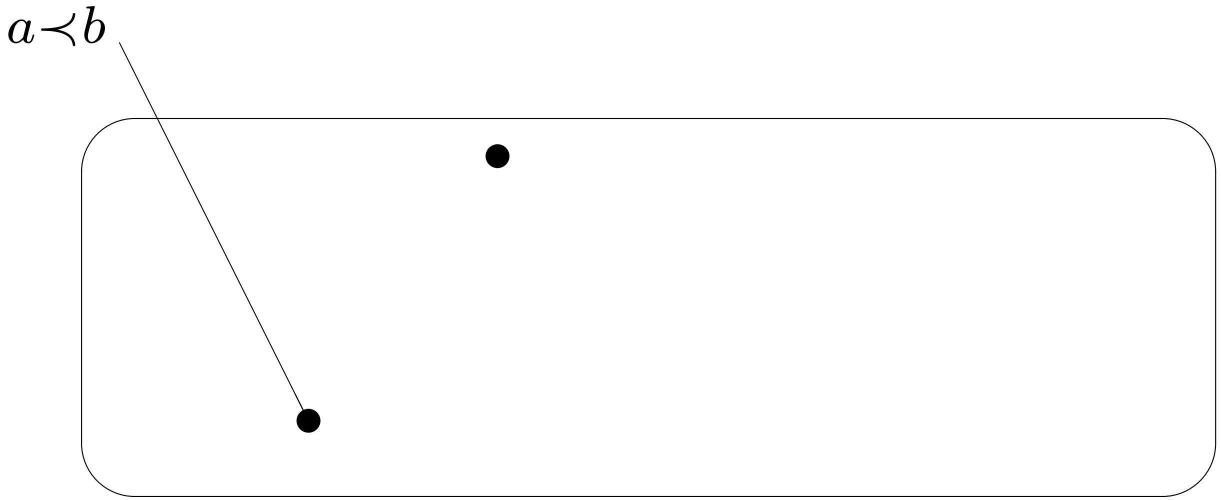
A Wellordering of the reals under AC:

We start with one number, say a.



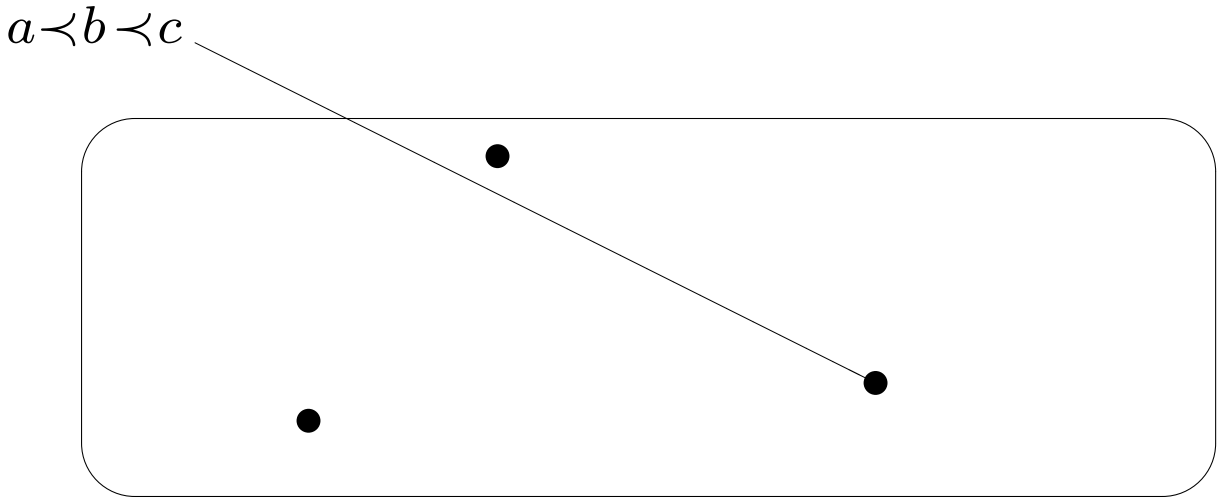
A Wellordering of the reals under AC:

Then we choose a number b from $\mathbb{R}\setminus\{a\}$,



A Wellordering of the reals under AC:

then c from $\mathbb{R}\setminus\{a,b\}$, ...



A Wellordering of the reals under AC:

In the end we will have ordered all reals.

$$a \prec b \prec c \prec d \prec e \prec f \prec \cdots$$

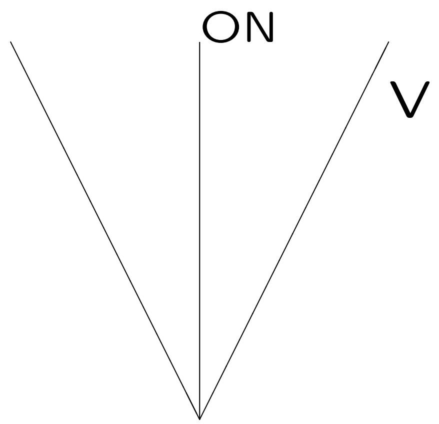
[redacted]

Ordinals:

We described the natural numbers as sets, where $n \in \mathbb{N}$ was formalized as $n = \{m \,|\, m < n\}$ and $m < n$ meant $m \in n$.

If we generalize this idea we get the Ordinal numbers. Formally an ordinal is a transitive, wellordered set. The collection of all ordinals is not a set, it is “to big.”

A picture of the universe:



We write ON for the class of ordinals and for our purposes we need only to know that the ordinals are wellordered by $<$.

A Wellordering of the reals under AC:

Let $f : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ be a choice function for the power set of the reals.

Define the function $F\colon \mathrm{ON} \to \mathbb{R}$ by transfinite recursion (here α and β denote ordinals):

$$F(\alpha) := \begin{cases} f\left(\mathbb{R}\setminus \bigcup\{F(\beta) \mid \beta < \alpha\}\right), \\ \text{undefined if } \bigcup\{F(\beta) \mid \beta < \alpha\} = \mathbb{R}. \end{cases}$$

If we now say $a \prec b$ iff $a = F(\alpha)$, $b = F(\beta)$ and $\alpha < \beta$, then $\prec$ is a wellordering on $\mathbb{R}$.

Infinite Games:

Let X be a collection of sequences $\langle x_i \mid i \in \mathbb{N} \rangle$ of natural numbers. Call X the payoff set.

A round in the game G_X is a sequence $x = \langle x_i \mid i \in \mathbb{N} \rangle$ of natural numbers, where the elements with even index $x_{\mathrm{I}}(i) := x_{2i}$ represent the moves of player I and those with odd index $x_{\mathrm{II}}(i) := x_{2i+1}$ the moves of player II. If $x \in X$ then player II wins, otherwise player I.

I	x_0		x_2		x_4		x_6		$\dots \quad (= x_{\mathrm{I}})$
II		x_1		x_3		x_5		x_7	$\dots \quad (= x_{\mathrm{II}})$

The Axiom of Determinacy:

An infinite game G_X is determined, if one of the players has a winning strategy, i.e. if there is a function that tells one player what to do next, depending on the previous moves, s.t. in the end that player wins the game regardless of the moves of his opponent.

So a strategy for player I would be a function of the form

$$f : \bigcup\{\mathbb{N}^{2n} \,|\, n \in \mathbb{N}\} \to \mathbb{N}.$$

The Axiom of Determinacy now is simply the statement

(AD) All infinite games are determined.

Countable Choice under AD:

AD is not compatible with full choice, but it implies countable choice for collections of sequences from $\mathbb{N}$.

Let $M = \{S_j \mid j \in \mathbb{N}\}$ be a countable collection of nonempty sets of such sequences:

Countable many S_j

S_1	S_2	S_3	$\cdots$	$\cdots$
$x_1 = \langle x_1(i) \mid i \in \mathbb{N} \rangle$	$y_1 = \langle y_1(i) \mid i \in \mathbb{N} \rangle$	$z_1 = \langle z_1(i) \mid i \in \mathbb{N} \rangle$	$\cdots$	$\cdots$
$x_2 = \langle x_2(i) \mid i \in \mathbb{N} \rangle$	$y_2 = \langle y_2(i) \mid i \in \mathbb{N} \rangle$	$z_2 = \langle z_2(i) \mid i \in \mathbb{N} \rangle$	$\cdots$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\cdots$
$x_\alpha = \langle x_\alpha(i) \mid i \in \mathbb{N} \rangle$	$y_\alpha = \langle y_\alpha(i) \mid i \in \mathbb{N} \rangle$	$z_\alpha = \langle z_\alpha(i) \mid i \in \mathbb{N} \rangle$	$\cdots$	
$\vdots$	$\vdots$	$\vdots$	$\cdots$	

Countable Choice under AD:

We now construct the payoff set X from M:

For each $j \in \mathbb{N}$ we define X_j to be the collection of all sequences x that start with j and where $x_{\mathrm{II}} \in S_j$. Let X be the union of the X_j.

$$X = \bigcup \{X_j \mid j \in \mathbb{N}\}$$

X_1	X_2	X_3	$\cdots$	$\cdots$
$\langle 1, x_1(1), \square, x_1(2), \square, \cdots \rangle$	$\langle 2, y_1(1), \square, y_1(2), \square, \cdots \rangle$	$\cdots$	$\cdots$	$\cdots$
$\vdots$	$\vdots$	$\cdots$	$\cdots$	$\cdots$
$\langle 1, x_2(1), \square, x_2(2), \square, \cdots \rangle$	$\langle 2, y_2(1), \square, y_2(2), \square, \cdots \rangle$	$\cdots$	$\cdots$	$\vdots$
$\vdots$	$\vdots$	$\cdots$	$\cdots$	
$\langle 1, x_3(1), \square, x_3(2), \square, \cdots \rangle$	$\langle 2, y_3(1), \square, y_3(2), \square, \cdots \rangle$	$\cdots$	$\vdots$	
$\vdots$	$\vdots$	$\vdots$		

Countable Choice under AD:

Then player I cannot have a winning strategy in the game G_X: Anything he does after his first move j has no consequences on the outcome, and player II only needs to play one of the sequences from the set S_j to refute him.

So by AD player II must have a winning strategy.

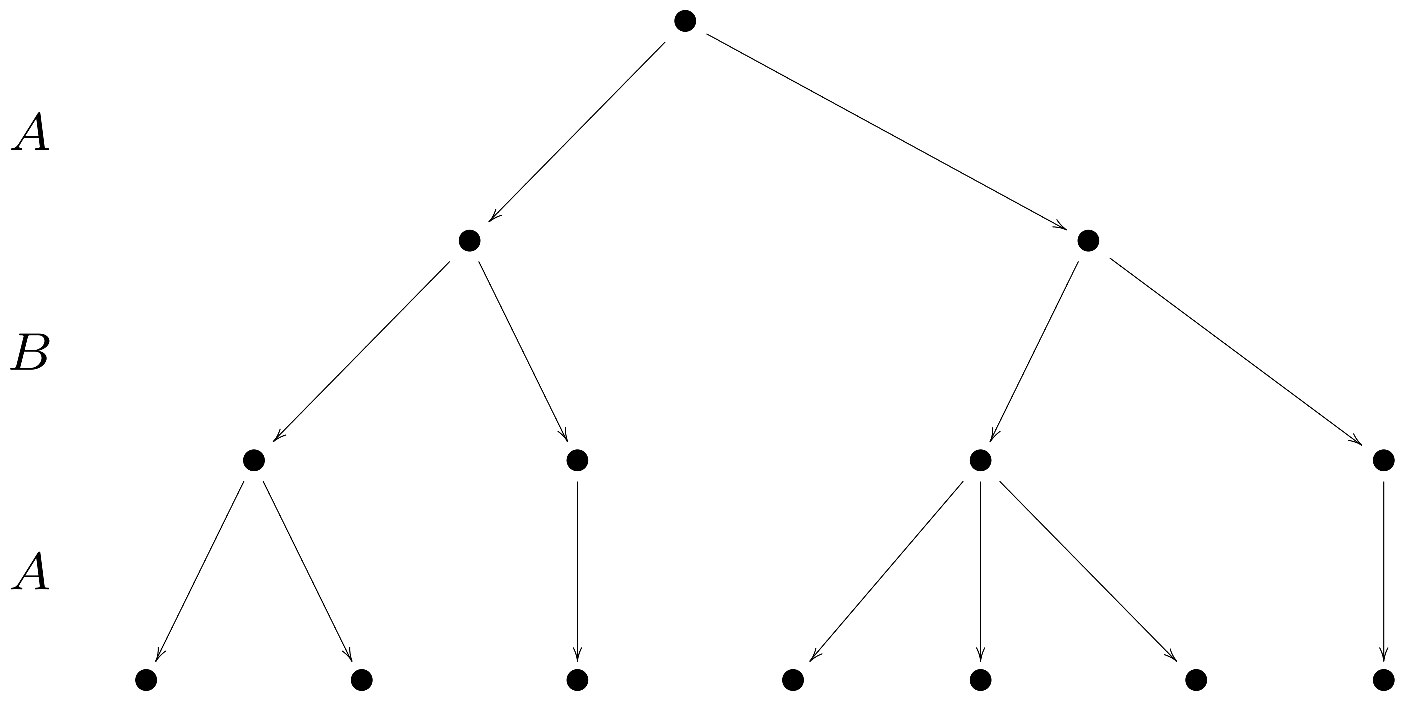
Let τ be II's winning strategy and let $(x * \tau)_{\mathrm{II}}$ be the sequence of player II's moves in the game where player I plays x and player II plays according to τ.

Then $F(S_j) := (\langle j \mid i \in \mathbb{N} \rangle * \tau)_{\mathrm{II}}$ defines a choice function for M.

Finite games:

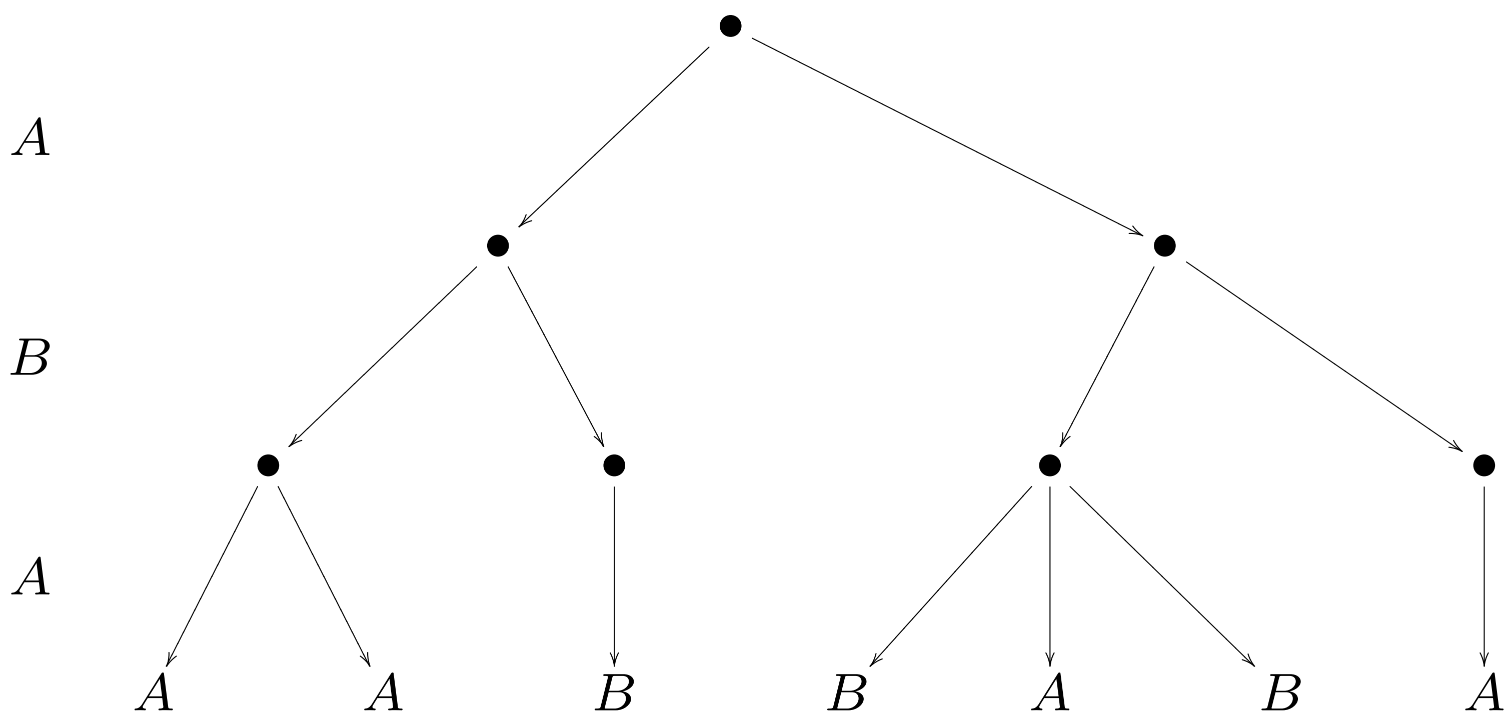
Every finite extensive game of perfect information that is a Win/Lose-game is determined. To see which of the two players A and B has the winning strategy, we use a simple backtracking method:

Let this tree represent such a game, the players take turns and player A starts:



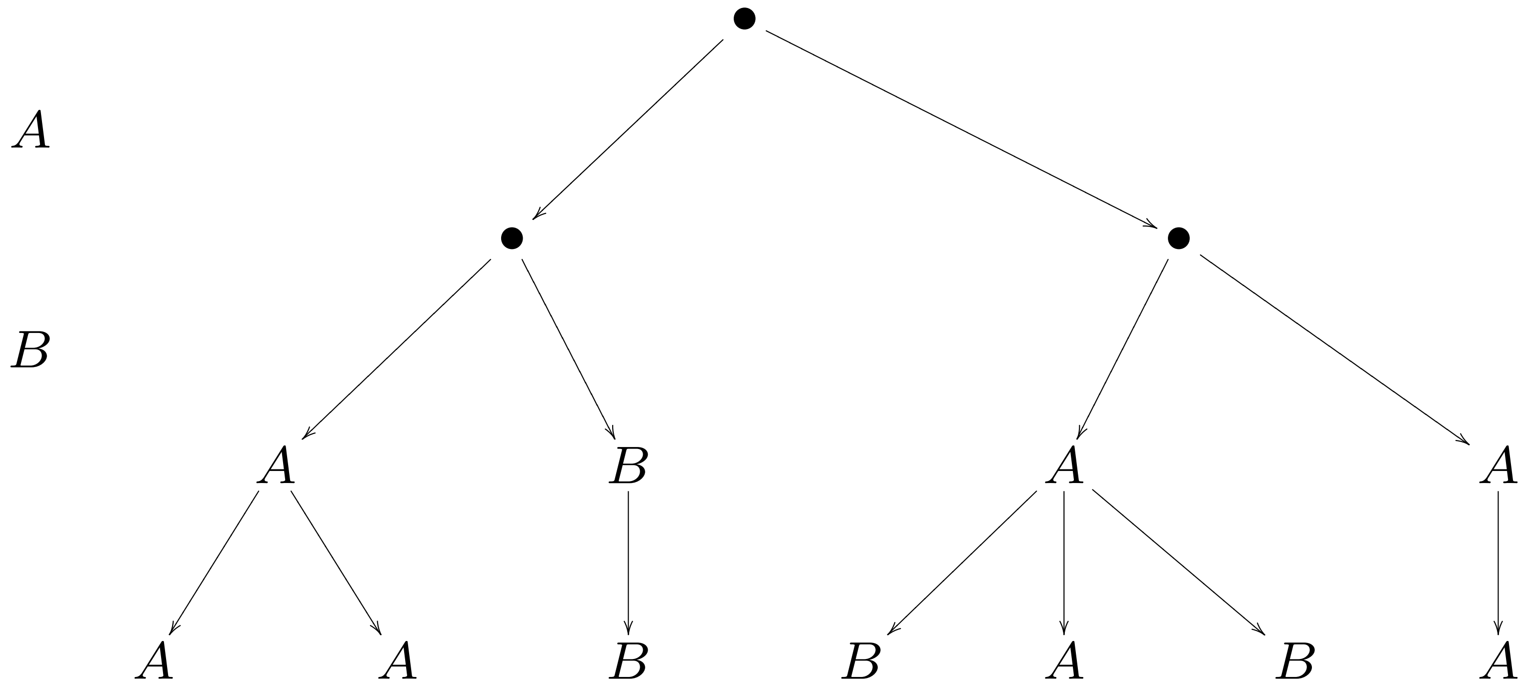
Winning strategies for finite games:

First we label the end-nodes:



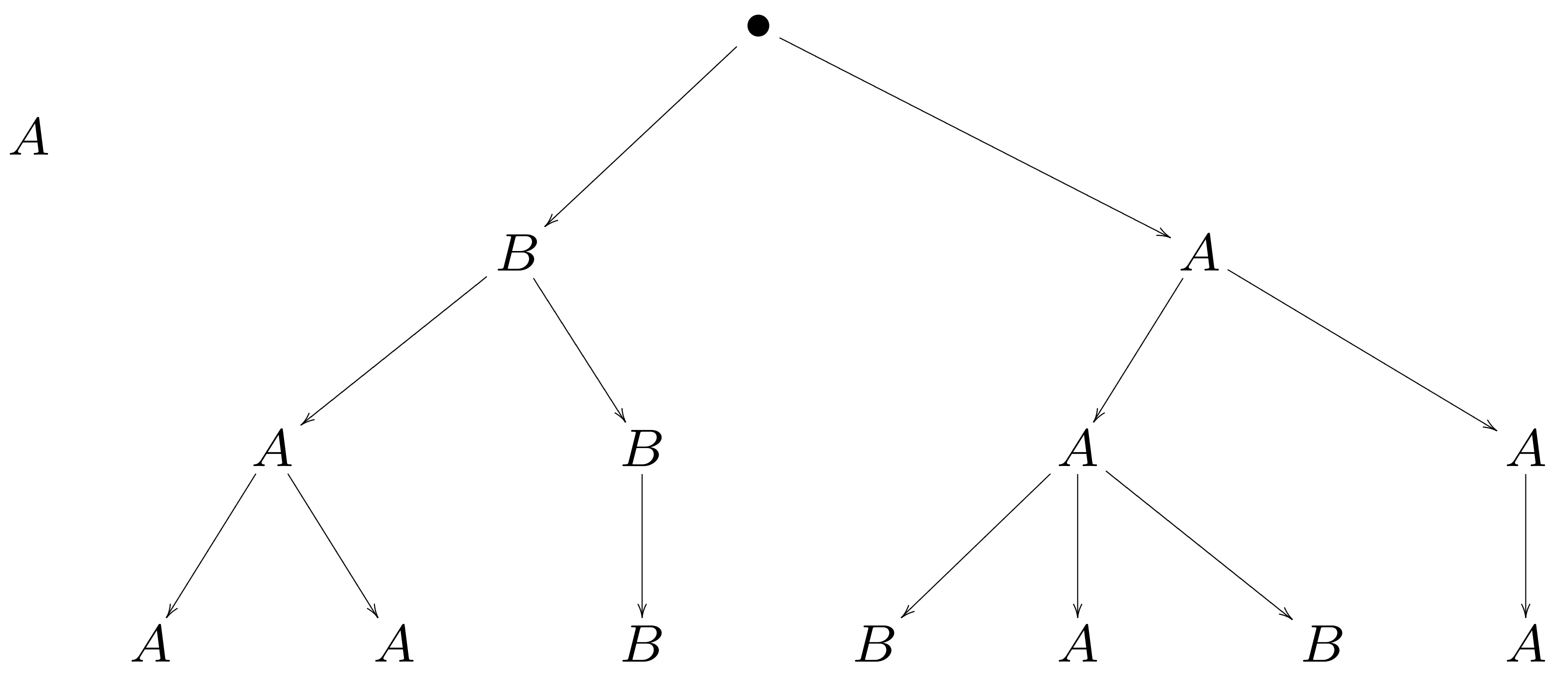
Winning strategies for finite games:

The last move is A's, we label the previous nodes accordingly:



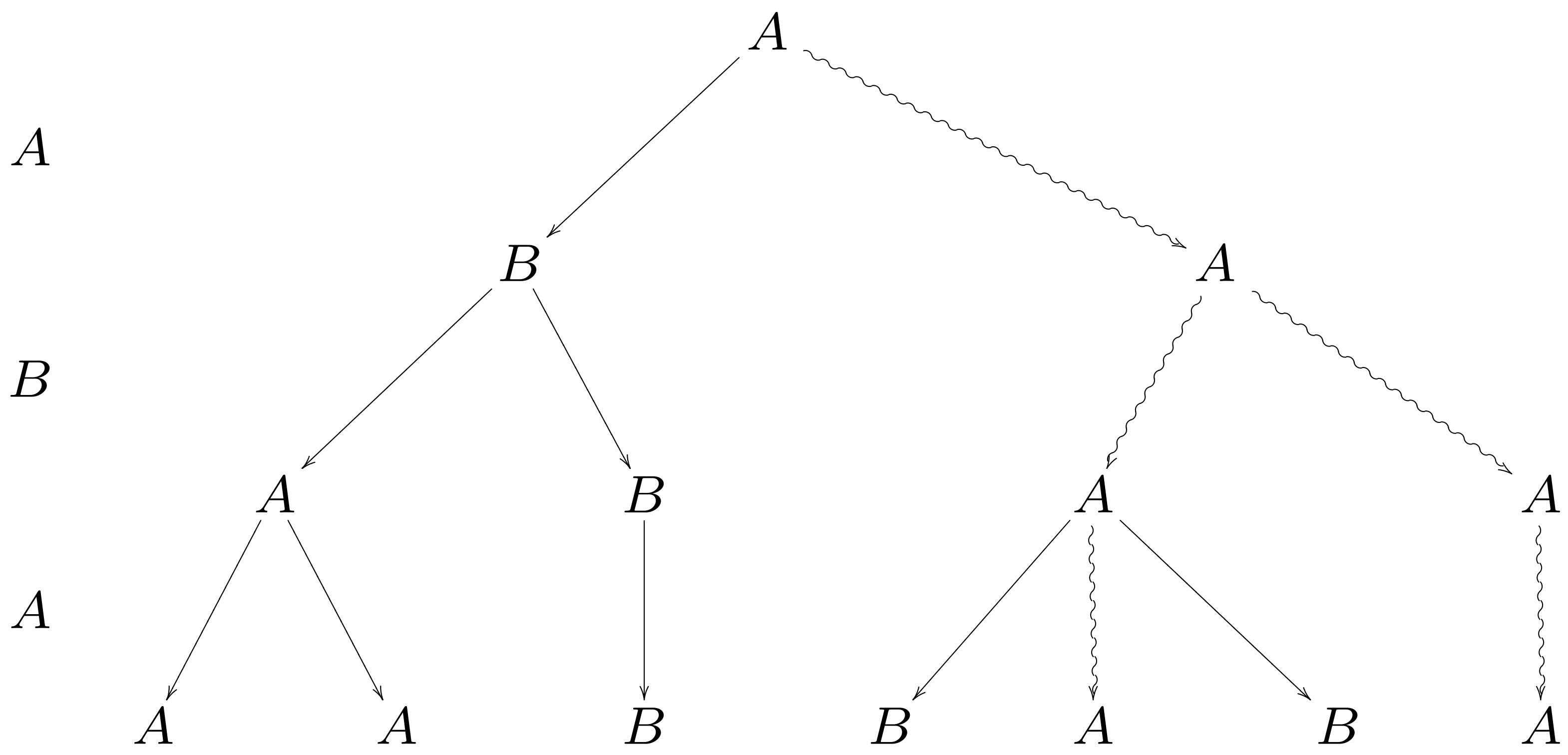
Winning strategies for finite games:

Now it is B's turn:



Winning strategies for finite games:

And in the end we see that player A has a winning strategie:



Similarities between AC and AD:

- Both axioms grant the existence of functions.

- Both imply restricted versions of the other.

- Both are extensions of finite principles.

- Both are consistent with ZF (under some assumptions).

- They contradict each other.

and Differences:

- Under AC exists a wellordering of the reals, but not under AD.
- Under AD every subset of the reals is Lebesgue measurable, not so under AC.
- Under AC every set can be wellordered, but not under AD.
- Under AD infinite partition properties hold, but not under AC.
- $\cdots$