AD and descriptions for cardinals below $\aleph_{\varepsilon_0}$

Modern Set Theory is the study of all possible worlds[1].

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AD and descriptions
for
cardinals below $\aleph_{\varepsilon_0}$
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Proposed in 1962 by Mycielski and Steinhaus, the Axiom of Determinacy has become one of the most interesting possible additions to ZF. It has profound consequences in analysis and descriptive set theory. Most prominent, under $\mathsf{ZF}+\mathsf{AD}$ all subsets of the reals are Lebesgue measurable, have the Baire property and are either countable or contain a perfect subset. Although AD contradicts AC, countable choice holds under AD and DC is consistent with it. And on the other hand, Borel sets are determined under ZFC, and if one includes large cardinal assumptions, then $\mathrm{L}(\mathbb{R})$ is a model of both AD and DC.
But most intriguing is the pattern of large cardinals that is generated by AD. Many types of large cardinals defined in terms of existence of measures and/or infinite partition properties can be proven to exist under AD. For example, $\aleph_1$ is not only measurable under AD, but has also the strong partition property. Early proofs of these results are due to Solovay, Martin, Kunen, and Kleinberg. Since 1985, Steve Jackson has produced many results of this type along with a general theory, called description theory.
This talk is a report about a work in progress. My work is part of a project to understand the combinatorial structure of cardinals under AD in more detail. Other participants of this project are Steve Jackson and Benedikt Löwe.

[1] J.M. Henle "Researches into the world of $\kappa \to (\kappa)^\kappa$", Annals of Mathematical Logic 17 (1979), 151–169.

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Definitions:

$$\boldsymbol{\delta}^1_n \quad := \quad \sup\{|\rho| \mid \rho \in \boldsymbol{\Delta}^1_n \text{ and } \rho \text{ is a prewellordering}\}.$$

$$\kappa \to (\kappa)^\lambda \quad :\Leftrightarrow \quad \text{For all } P : [\kappa]^\lambda \to 2 \text{ exists } A \subseteq \kappa \text{ s.t. } |A| = \kappa \text{ and } |P''[A]^\lambda| = 1.$$

$$\kappa \to (\kappa)^{<\lambda} \quad :\Leftrightarrow \quad \kappa \to (\kappa)^\alpha \text{ holds for all } \alpha < \lambda.$$

If $\kappa \to (\kappa)^\kappa$ holds, we say that κ has the **Strong Partition Property**.

Let me start with some definitions and notational conventions. The projective ordinals $\boldsymbol{\delta}^1_n$ are projective analogs of Θ, the largest ordinal, s.t. a surjection from the reals onto this ordinal exists. $\boldsymbol{\delta}^1_n$ is the supremum of the lengths of $\boldsymbol{\Delta}^1_n$-prewellorderings of the reals.
Also let me remind you about the following notation of partition properties:
"κ goes to κ to the λ" denotes "For all 2-partitions of the set of increasing functions from λ to κ exists a homogeneous set of size κ."
And "κ goes to κ to the less than λ" of course stands for "κ goes to κ to the α holds for all α less than λ."
We say that an infinite cardinal has the strong Partition Property if "κ goes to κ to the κ" holds for this cardinal. The strong partition propertie implies the existence of large cardinals, for example the measurability of the κ.
A theorem of Erdös/rado states that under AC $\kappa \to (\kappa)^\omega$ holds for no infinitive cardinal κ, so the strong Partition Property contradicts AC. But without AC there may exist many cardinals with infinite partition properties, as we will see.

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Facts[AD]**:**

- $\boldsymbol{\delta}^1_1 = \aleph_1$ (class.), $\boldsymbol{\delta}^1_2 = \aleph_2$ (Martin).
- $\boldsymbol{\delta}^1_3 = \aleph_{\omega+1}$, $\boldsymbol{\delta}^1_4 = \aleph_{\omega+2}$ (Martin/Solovay).
- $\boldsymbol{\delta}^1_1 \to (\boldsymbol{\delta}^1_1)^{\boldsymbol{\delta}^1_1}$ (Martin).
- $\boldsymbol{\delta}^1_{2n+1} = \aleph_{\mathrm{w}(2n+1)+1}$ (Jackson), so $\sup\{\boldsymbol{\delta}^1_n \mid n \in \omega\} = \aleph_{\varepsilon_0}$.
  (Here $\mathrm{w}(1) := 0$, $\mathrm{w}(n+1) := \omega^{\mathrm{w}(n)}$.)
- $\boldsymbol{\delta}^1_{2n+1} \to (\boldsymbol{\delta}^1_{2n+1})^{\boldsymbol{\delta}^1_{2n+1}}$ (Jackson).

From now on we will work under $\mathsf{ZF} + \mathsf{AD}$. A classical result is $\boldsymbol{\delta}^1_1 = \omega_1$ Martin showed $\boldsymbol{\delta}^1_2 = \aleph_2$, by a result of Kechris the projective ordinals with even index are in fact generally the successors of those with odd index. Martin and Solovay proved $\boldsymbol{\delta}^1_3 = \aleph_{\omega+1}$ and Martin also showed that the strong partition property for $\delta 3$ holds under AD.
So, what is $\boldsymbol{\delta}^1_5$? This question became the 5th Victoria Delfino problem and was finally answered in 1983 by Steve Jackson. In his PhD thesis he computed $\boldsymbol{\delta}^1_5$ to be $\aleph_{\omega^{\omega^\omega}+1}$. In the following years he generalized the methods and ideas behind this proof to deal with all projective ordinals. The result, $\boldsymbol{\delta}^1_{n+1} = \aleph_{w(2n+1)+1}$, where w is defined recursively by $w(0) = 1$ and $w(n+1) = \omega^{w(n)}$ was proven inductively, one main ingredient of the induction was to prove the strong partition property for the projective ordinal in each step. So the supremum of the projective ordinals is $\aleph_{\varepsilon_0}$ and all of them have the strong partition property.

The picture so far looks like this: (Slide 4)

Slide 4

The projective ordinals under AD:

$$
\begin{array}{cccccccccccccc}
\aleph_0 & \aleph_1 & \aleph_2 & \cdots & \aleph_\omega & \aleph_{\omega+1} & \aleph_{\omega+2} & \cdots & \aleph_{\omega^{\omega^\omega}} & \aleph_{\omega^{\omega^\omega}+1} & \aleph_{\omega^{\omega^\omega}+2} & \cdots & \aleph_{\omega^{\omega^{\omega^{\omega^\omega}}}+1} & \aleph_{\omega^{\omega^{\omega^{\omega^\omega}}}+2} \\
 & \shortparallel & \shortparallel & & & \shortparallel & \shortparallel & & & \shortparallel & \shortparallel & & \shortparallel & \shortparallel \\
 & \boldsymbol{\delta}^1_1 & \boldsymbol{\delta}^1_2 & & & \boldsymbol{\delta}^1_3 & \boldsymbol{\delta}^1_4 & & & \boldsymbol{\delta}^1_5 & \boldsymbol{\delta}^1_6 & & \boldsymbol{\delta}^1_7 & \boldsymbol{\delta}^1_8
\end{array}
$$

Jacksons method, which I will call Description Analysis, uses an Upper Bound/Lower Bound argument to compute the value of a projective ordinal. For the lower bound, a cofinal sequence of ultrapowers w.r.t. certain canonical measures is embedded. While this suffices to compute the projective ordinals, it is not enough to give descriptions for all cardinals below $\aleph_{\varepsilon_0}$.

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Description Analysis	Jackson/Khafizov Analysis
S.Jackson A Computation of $\boldsymbol{\delta}^1_5$ Mem. Amer. Math. Soc. 140 (1999)	S.Jackson + F.Khafizov Descriptions and Cardinals below $\boldsymbol{\delta}^1_5$ *to appear*
Upper/lower bounds for projective ordinals	Complete analysis of cardinals $< \boldsymbol{\delta}^1_5$

For cardinals below $\boldsymbol{\delta}^1_5$ a finer analysis was done in 1995 by Steve Jackson and Farid Khafizov. If $\aleph_\alpha$ is a crdinal below $\boldsymbol{\delta}^1_5$, then the α canonically describes an order mesure, whose ultrapower is exactly $\aleph_\alpha$. Among other results, this Jackson/Khafizov Analysis gives us full knowledge of the cofinalities of cardinals below $\boldsymbol{\delta}^1_5$.

How can we use this knowledge?

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Theorem[ZF]**(Kleinberg):**
Let κ be a cardinal s.t. $\kappa \to (\kappa)^\kappa$, let μ be a normal measure on κ and let $\kappa_1 := \kappa$ and $\kappa_{n+1} := \kappa_n^\kappa/\mu$. Then

1. κ_1 and κ_2 are measurable.
2. For all $n \geq 2$, $\mathrm{cf}(\kappa_n) = \kappa_2$.
3. κ_n is a Jónsson cardinal.
4. if $\kappa^\kappa/\mu = \kappa^+$, then $\kappa_{n+1} = \kappa_n^+$ for all $n \in \omega$.

This Theorem by Eugene Kleinberg, 1977, gives us a methode to generate a sequence of Jónsson cardinals by an iterated ultrapower construction. Also, if the length of the ultrapower of κ is its successor, we know the values of all cardinals in the Kleinberg sequence. The theorem is true under ZF, but to apply it we need the strong partition property of κ and the existence of normal measures.
But now we can put our knowledge about the projective ordinals under AD to good use and calculate some Kleinberg sequences.

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Theorem[AD]**:**

- $\aleph_{n+1}$ is Jónsson.
- $\aleph_{\omega+n+1}$ is Jónsson.
- In general, $\aleph_{w(2m+1)+n+1}$ is Jónsson for all $n, m \in \omega$.

Theorem[AD]**(B.Löwe 2002):**

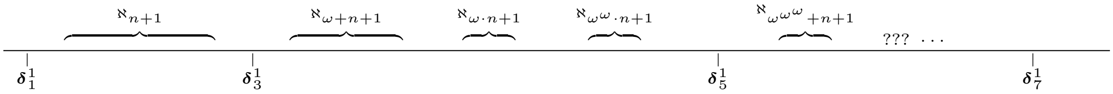
- $\aleph_{\omega\cdot n+1}$ is Jónsson.
- $\aleph_{\omega^\omega\cdot n+1}$ is Jónsson.

Using Jacksons results about the projective ordinals and the fact that the ultrapower of $\boldsymbol{\delta}^1_{2n+1}$ w.r.t. the ω-cofinal measure on it is $\boldsymbol{\delta}^1_{2n+2}$, its successor, we immediately get these sequences of Jónsson cardinals.
Benedict Löwe used the finer Jackson/Khafizov Analysis to compute the Kleinberg sequences that correspond to the to other normal measures on $\boldsymbol{\delta}^1_3$. In order to do this, he proved a technical computational lemma, the Ultrapower Shifting Lemma, that gives a bound for the length of the ultrapowers.

So now the pattern looks like this: (Slide 8)

Slide 8

The Pattern so far:



Ingredients:

- Kleinbergs Theorem
- Technical Computation (Ultrapower Shifting Lemma, Löwe)
- Full Jackson/Khafizov Analysis of cardinals $< \boldsymbol{\delta}^1_5$

Between two neighboring projective ordinals are Kleinberg sequences of Jónsson cardinals, stemming from normal measures. Right now this stops, except for the ω_0-cofinal measure, at $\boldsymbol{\delta}^1_5$. My goal is to generalize the Jackson/Khafizov Analysiss to the higher projective ordinals. Since Kleinbergs theorem and the Ultrapower Shifting Lemma do not depend on the level in the hierarchy of projective ordinals, this would enable us to compute the Kleinberg sequences that come from the other normal measures on the projective ordinals.
Another question is: What about the cardinals in the gaps between the Kleinberg sequences?

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First Steps towards a general Jackson/Khafizov Analysis allow us to deal with taking sums of measures.

Theorem[AD]**:** (Using some technical assumptions)

- $\aleph_{\omega^{\omega^\omega}+\omega\cdot n+1}$ is Jónsson.

First steps towards a general Jackson/Khafizov Analysis allow us to deal with ultrapowers coming from sums of measures. Under some technical assumptions, for example that the ultrapowers in question are cardinals, we were able to get this result.

I thank you for attending my talk.