A simple inductive argument to compute more Kleinberg sequences under the Axiom of Determinacy

Colloqium Logicum 2004, Heidelberg

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This research has been supported by the DFG-NWO Bilateral Cooperation Project KO 1353/3-1/DN 61-532 Determiniertheitsaxiome, Infinitäre Kombinatorik und ihre Wechselwirkungen (2003-2006; Bold, Koepke, Löwe, van Benthem)

Definitions and Notations:

The iterated successor operation on cardinals κ is defined by:

1. $\kappa^{(0)} = \kappa$,
2. $\kappa^{(\alpha+1)} = (\kappa^{(\alpha)})^{+}$ for all ordinals α, and
3. $\kappa^{(\lambda)} = \bigcup\{\kappa^{(\alpha)}\,;\,\alpha \in \lambda\}$ for limit ordinals λ.

A cardinal κ is a **strong partition cardinal** (in Erdős arrow notation: $\kappa \to (\kappa)^{\kappa}$) if for every partition $F : [\kappa]^{\kappa} \to 2$ exists a homogeneous set $H \subseteq \kappa$ of cardinality κ, *i.e.* Card$(F``[H]^{\kappa}) = 1$. Note that the existence of a strong partition cardinal violates the Axiom of Choice.

If μ is a measure on κ and α is an ordinal, then we write α^{κ}/μ for the (Mostowski-collapse of the) ultrapower of α with respect to μ. Under ZF + DC the ultrapower α^{κ}/μ is an ordinal.

Theorem [Kleinberg]: Assume $\mathsf{ZF} + \mathsf{DC}$. Let κ be a strong partition cardinal cardinal, let μ be a normal measure on κ and let $\kappa_1^\mu := \kappa$ and $\kappa_{n+1}^\mu := (\kappa_n^\mu)^\kappa/\mu$. Then

1. κ_1^μ and κ_2^μ are measurable,
2. for all $n \geq 2$, $\mathrm{cf}(\kappa_n^\mu) = \kappa_2^\mu$,
3. all κ_n^μ are Jónsson cardinals, and
4. $\sup\{\kappa_n^\mu\,;\, n \geq 1\}$ is a Rowbottom cardinal.
5. Moreover, if $\kappa^\kappa/\mu = \kappa^+$, then $\kappa_{n+1}^\mu = (\kappa_n^\mu)^+$ for all $n \in \omega$.

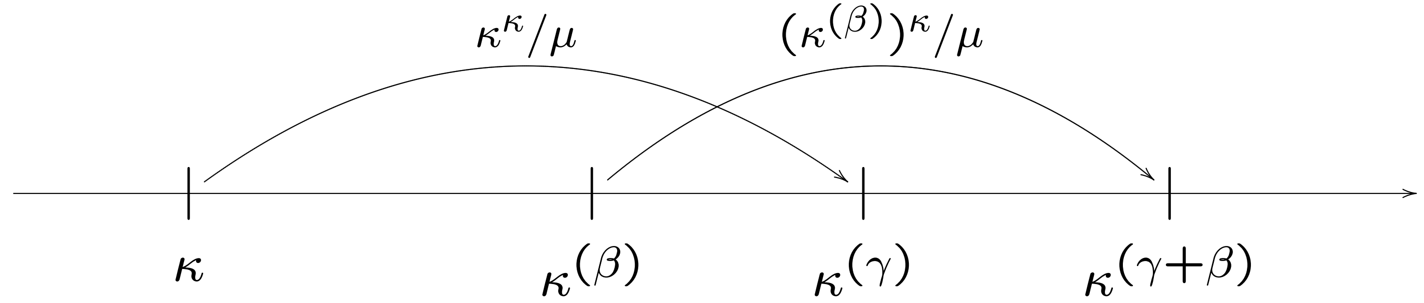
We call the sequence $\langle \kappa_n^\mu\,;\, n \geq 1\rangle$ the **Kleinberg sequence derived from** μ.

A proof of this theorem can be found in: Eugene M. **Kleinberg**, Infinitary Combinatorics and the Axiom of Determinateness, Springer-Verlag 1977 [Lecture Notes in Mathematics 612] .

Theorem [Ultrapower Shifting Lemma]: Assume ZF+DC. Let β and γ be ordinals and let μ be a κ-complete ultrafilter on κ with $\kappa^\kappa/\mu = \kappa^{(\gamma)}$. If for all cardinals $\kappa < \nu \leq \kappa^{(\beta)}$

- either ν is a successor and cf$(\nu) > \kappa$,
- or ν is a limit and cf$(\nu) < \kappa$,

then $(\kappa^{(\beta)})^\kappa/\mu \leq \kappa^{(\gamma+\beta)}$.



A proof of this theorem can be found in: Benedikt **Löwe**, Kleinberg Sequences and partition cardinals below $\boldsymbol{\delta}^1_5$, **Fundamenta Mathematicae** 171 (2002), p. 69–76.

An abstract combinatorial computation

Lemma 1: Assume ZF+DC. Let $\kappa < \lambda$ be cardinals, μ a measure on κ and $\text{cf}(\lambda) > \kappa$. Then $\text{cf}(\lambda^\kappa/\mu) = \text{cf}(\lambda)$.

Proof: "$\leq$": For $\alpha < \lambda$ let $c_\alpha : \kappa \to \lambda$ be the constant function $c_\alpha(\xi) = \alpha$. We shall show that $\{[c_\alpha]_\mu\,;\,\alpha \in \lambda\}$ is cofinal in λ^κ/μ: Let $f \in \lambda^\kappa$ be arbitrary. Since $\text{cf}(\lambda) > \kappa$, the range of the function f is bounded in λ, *i.e.*, there is an $\alpha^* \in \lambda$ such that $\{f(\xi)\,;\,\xi \in \kappa\} \subseteq \alpha^*$. Then $[f]_\mu < [c_{\alpha^*}]_\mu$.

"$\geq$": Now let $X \subseteq \lambda^\kappa/\mu$ be a cofinal subset. If $\xi \in X$, there is some $\alpha \in \lambda$ such that $\xi \leq [c_\alpha]_\mu$ by the above argument. Let α_ξ be the least such ordinal. We claim that $A := \{\alpha_\xi\,;\,\xi \in X\}$ is a cofinal subset of λ: Let $\gamma \in \lambda$ be arbitrary. Since X was cofinal, pick some $\xi_\gamma \in X$ such that $\xi_\gamma > [c_\gamma]_\mu$. But then, $\alpha_{\xi_\gamma} \in A$ with $\alpha_{\xi_\gamma} > \gamma$. So, A is cofinal in λ. But $\text{Card}(A) \leq \text{Card}(X)$, so $\text{cf}(\lambda) \leq \text{cf}(\lambda^\kappa/\mu)$. □

An abstract combinatorial computation

Theorem 1: Assume ZF+DC. Let κ be a strong partition cardinal and μ_0 and μ_1 be normal ultrafilters on κ with $\kappa^\kappa/\mu_0 = \kappa^+$ and $\kappa^\kappa/\mu_1 = \kappa^{(\omega+1)}$. For all $\beta < \omega^2$, assume that $(\kappa^{(\beta)})^\kappa/\mu_1$ is a cardinal.

Then for all $\xi < \omega^2$, the following equalities hold:

1. $(\kappa^{(\xi)})^\kappa/\mu_1 = \kappa^{(\omega+1+\xi)}$, and
2. $\mathrm{cf}(\kappa^{(\xi+1)}) = \begin{cases} \kappa^+ & \text{if } \xi \text{ is a successor or zero, or} \\ \kappa^{(\omega+1)} & \text{if } \xi > 0 \text{ is a limit.} \end{cases}$

Proof: By Kleinberg's Theorem (1.), (2.) and (5.), we have

$$\mathrm{cf}(\kappa^{(n+1)}) = \kappa^{+}$$

for $n \in \omega$. Also, for all limit ordinals $\lambda < \omega^2$, the cofinality of $\kappa^{(\lambda)}$ is ω. We denote these facts by (IH_*).

We proceed by induction on ξ with the induction hypothesis:*

$$(\mathrm{IH}_\xi) \left[\begin{array}{l} \text{For all } \alpha \leq \xi \text{, the following two conditions hold:} \\ 1.\ (\kappa^{(\alpha)})^{\kappa}/\mu_1 = \kappa^{(\omega+1+\alpha)}, \\ 2.\ \mathrm{cf}(\kappa^{(\omega+1+\alpha)}) := \begin{cases} \omega & \text{if } \alpha > 0 \text{ is a limit,} \\ \kappa^{+} & \text{if } \alpha \text{ is 1 or a double successor, or} \\ \kappa^{(\omega+1)} & \text{if } \alpha \neq 1 \text{ is zero or a single successor.} \end{cases} \end{array} \right.$$

*An ordinal γ is a double successor is there is some δ such that $\gamma = \delta + 2$. An ordinal is a single successor if it is the successor of a limit ordinal.

Proof(cont.):

By assumption, $(\kappa^{(0)})^\kappa/\mu_1 = \kappa^\kappa/\mu_1 = \kappa^{(\omega+1)}$ and from Kleinbergs Theorem (1.), we know that this is a regular cardinal, so (IH_0) holds.

For the successor step $\xi \mapsto \xi + 1$ assume that (IH_ξ) holds. Let us look at the Ultrapower Shifting Lemma with $\gamma = \omega + 1$ and $\beta = \xi + 1$. Since $\xi < \omega^2$, we have $\xi + 1 < \omega + 1 + \xi$, so (IH_ξ) and (IH_*) allows us to apply the Lemma and get:

$$\begin{aligned}
\kappa^{(\omega+1+(\xi+1))} &\geq (\kappa^{(\xi+1)})^\kappa/\mu_1 && \text{(Ulrapower Shifting Lemma)}\\
&> (\kappa^{(\xi)})^\kappa/\mu_1 \\
&= \kappa^{(\omega+1+\xi)}. && (\mathrm{IH}_\xi)
\end{aligned}$$

Since $(\kappa^{(\xi+1)})^\kappa/\mu_1$ is a cardinal (by assumption) lying in the interval between $\kappa^{(\omega+1+\xi)}$ and its successor, we get

$$(\kappa^{(\xi+1)})^\kappa/\mu_1 = \kappa^{(\omega+1+(\xi+1))}.$$

Proof(cont.):

We shall now compute the cofinality of $\kappa^{(\omega+1+(\xi+1))}$ in order to check that ($\mathrm{IH}_{\xi+1}$) holds:

Case 1: $\xi < \omega$. In this case, $\mathrm{cf}(\kappa^{(\xi+1)}) = \kappa^{+} > \kappa$ by (IH_*). So, we can apply Lemma 1 to $\lambda := \kappa^{(\xi+1)}$. Thus

$$\begin{aligned} \mathrm{cf}(\kappa^{(\omega+1+(\xi+1))}) &= \mathrm{cf}((\kappa^{(\xi+1)})^{\kappa}/\mu_1) \\ &= \mathrm{cf}(\kappa^{(\xi+1)}) && \text{(Lemma 1)} \\ &= \kappa^{+}. && (\mathrm{IH}_*) \end{aligned}$$

Proof(cont.):

Case 2: $\omega \leq \xi < \omega^2$. In this case, there is an ordinal $\alpha < \xi$ such that $\xi + 1 = \omega + 1 + \alpha$, and the following equivalences hold:

$$(*) \left[\begin{array}{ccc} \alpha \text{ is 1 or a double successor} & \Longleftrightarrow & \xi \text{ is a successor,} \\ \alpha \neq 1 \text{ is zero or a single successor} & \Longleftrightarrow & \xi \text{ is a limit.} \end{array} \right.$$

Now, by (IH_ξ), we get that $\mathrm{cf}(\kappa^{(\xi+1)}) = \mathrm{cf}(\kappa^{(\omega+1+\alpha)}) > \kappa$. So, again applying Lemma 1 to $\lambda := \kappa^{(\xi+1)}$, we get

$$\begin{aligned} \mathrm{cf}(\kappa^{(\omega+1+(\xi+1))}) &= \mathrm{cf}((\kappa^{(\xi+1)})^\kappa/\mu_1) \\ &= \mathrm{cf}(\kappa^{(\xi+1)}) \qquad \text{(by Lemma 1)} \\ &= \mathrm{cf}(\kappa^{(\omega+1+\alpha)}), \end{aligned}$$

thus by $(*)$

$$\mathrm{cf}(\kappa^{(\omega+1+(\xi+1))}) = \begin{cases} \kappa^+ & \text{if } \xi \text{ is a successor, and} \\ \kappa^{(\omega+1)} & \text{if } \xi \text{ is a limit.} \end{cases}$$

Proof(cont.):

For the limit step, let $0 < \lambda < \omega^2$ be a limit ordinal. Note that this implies that for some $\alpha < \lambda$, we have that $\omega + \alpha = \lambda$. We now assume (IH_η) for $\eta < \lambda$, and write $(\mathrm{IH}_{<\lambda})$ for this assumption. In particular (since $\alpha < \lambda$), we know the cofinalities of all cardinals between κ and $\kappa^{(\omega+1+\alpha)} \geq \kappa^{(\omega+\alpha)} = \kappa^{(\lambda)}$. This allows us to apply the Ultrapower Shifting Lemma for $\gamma = \omega + 1$ and $\beta = \lambda$:

$$\begin{aligned} \sup\{\kappa^{(\omega+1+\eta)}\,;\,\eta<\lambda\} &= \sup\{(\kappa^{(\eta)})^\kappa/\mu_1\,;\,\eta<\lambda\} \quad (\mathrm{IH}_{<\lambda}) \\ &\leq (\kappa^{(\lambda)})^\kappa/\mu_1 \\ &\leq \kappa^{(\omega+1+\lambda)} \quad \text{(Ultrapower Shifting Lemma)} \\ &= \sup\{\kappa^{(\omega+1+\eta)}\,;\,\eta<\lambda\}. \end{aligned}$$

This establishes $(\kappa^{(\lambda)})^\kappa/\mu_1 = \kappa^{(\omega+1+\lambda)}$. The claim about the cofinality of $\kappa^{(\omega+1+\lambda)}$ is trivial for a limit ordinal $\lambda < \omega^2$. $\square$

Applications to infinitary combinatorics under AD

We now move to the applications of the abstract Theorem 1 under AD. If $\lambda < \kappa$ are regular cardinals, the λ-**cofinal measure on** κ is defined to be the filter generated by sets of the type

$$\{\alpha \in \kappa\,;\, \alpha \in C \ \&\ \mathsf{cf}(\alpha) = \lambda)\}$$

for some closed unbounded subset C of κ. We write $\mathcal{C}^{\lambda}_{\kappa}$ for this filter. The projective ordinals are defined as follows:

$$\boldsymbol{\delta}^1_n := \sup\{\alpha\,;\, \text{there is a } \mathbf{\Delta}^1_n \text{ prewellordering of } \mathbb{R} \text{ of length } \alpha\}.$$

Applications to infinitary combinatorics under AD

The following theorem is a summary of work due to Kleinberg, Kunen, Martin and Jackson:

Theorem 2: Assume $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}$.
Let $\mathrm{e}_0 := 0$ and $\mathrm{e}_{n+1} := \omega^{(\omega^{\mathrm{e}_n})}$.

1. If $\lambda < \boldsymbol{\delta}^1_{2n+1}$ is regular, then $\mathcal{C}^{\lambda}_{\boldsymbol{\delta}^1_{2n+1}}$ is a normal measure on $\boldsymbol{\delta}^1_{2n+1}$,
2. for all n, the ordinal ${\boldsymbol{\delta}^1_{2n+1}}^{\boldsymbol{\delta}^1_{2n+1}} / \mathcal{C}^{\omega}_{\boldsymbol{\delta}^1_{2n+1}} = \boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$,
3. $\boldsymbol{\delta}^1_{2n+1} = \aleph_{\mathrm{e}_n+1}$, and
4. $\boldsymbol{\delta}^1_{2n+1}$ is a strong partition cardinal.

Applications to infinitary combinatorics under AD

Theorem: Assume $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}$. Assume furthermore that

1. ${\boldsymbol{\delta}^1_{2n+1}}^{\boldsymbol{\delta}^1_{2n+1}} / \mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}} = \aleph_{\mathbf{e}_n+\omega+1}$, and that
2. for all $\xi < \omega^2$, the ordinal ${\aleph_{\mathbf{e}_n+\xi}}^{\boldsymbol{\delta}^1_{2n+1}} / \mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}}$ is a cardinal.

Then for each $m \in \omega$, the cardinal $\aleph_{\mathbf{e}_n+\omega\cdot m+1}$ is Jónsson, and $\aleph_{\mathbf{e}_n+\omega^2}$ is Rowbottom.

Proof: Let $\mu_0 := \mathcal{C}^{\omega}_{\boldsymbol{\delta}^1_{2n+1}}$ and $\mu_1 := \mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}}$ and let $\kappa_m := \kappa_m^{\mu_1}$ be the elements of the Kleinberg sequence derived from μ_1, *i.e.*, $\kappa_{m+1} = (\kappa_m)^{\kappa}/\mu_1$.

By Theorem 2 and the assumptions, all requirements of Theorem 1 are met, and so we can inductively read off the values of

$$\begin{aligned} \kappa_1 &= \boldsymbol{\delta}^1_{2n+1} = \aleph_{\mathbf{e}_n+1}, \\ \kappa_{m+1} &= (\kappa_m)^{(\omega+1)} \text{ (for } m \geq 1), \end{aligned}$$

and so

$$\kappa_{m+1} = \aleph_{\mathbf{e}_n+\omega\cdot m+1}.$$

Now the theorem follows directly from Kleinberg's Theorem.

$\square$.

`http://www.math.uni-bonn.de/people/logic/People/bold.html`

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