Dr. René Mboro

GRADUATE SEMINAR ON COMPLEX GEOMETRY (S4D3)

Summer term 2019
Wednesday 2:00-4:00 pm, N 0.007 - Neubau

The seminar is intended to be an introduction to complex geometry as presented mainly in [4]. We introduce complex manifolds, Kähler metrics on them and study the consequences of the existence of such metrics. Complex manifolds admitting a Kähler metric are called Kähler manifolds. Smooth complex projective varieties - which are the main objects studied by complex algebraic geometry -, that is, zero sets of families of polynomials in projective space form an important class of examples of Kähler manifolds. We will see how Hodge theory reveals the existence of strong constraints on the topology of a complex manifold which is Kähler. In particular, we will see that many complex manifolds cannot be endowed with a structure of a complex projective variety. The seminar ends with an introduction to variations of Hodge structure.
Prerequisities: We assume basic concepts of differential geometry, such as differentiable manifolds, differential forms and de Rham cohomology. Some of this material is collected in [4, Appendix A]. The participants who are not familiar with sheaf cohomology should have a look at [4, Appendix B]. The participants should also know the theory of holomorphic functions in one variable and have a look at [4, Section 1.1] where some principal facts about the local theory of holomorphic functions in several variables are presented.

Our program follows mainly [4], with some material coming from [5]. Another source is [1]. More advanced references which go deeper into the subject are [2],[3].

Should you be interested in the seminar, please send an email to René Mboro (rmboro@uni-bonn.de) naming two or three of the talks you would be interested to give. The talks will be distributed by early March.
Every speaker is to contact René Mboro at least two weeks before the talk to discuss his topic with him. He will also be the person to contact should you encounter any problems in yours or anyone elses talk or for references.
Each talk is 50 min and should be prepared accordingly. Questions during and after the talks are welcome. The speaker is also supposed to solve a set of assigned exercises (related to the topic of his talk). Exercise sheet will be provided at each session; they are optional (i.e. for people eager to test what they have learned from the talks).

0.1. Complex manifolds and holomorphic vector bundles (3.4.19).

0.1.1. *Complex manifolds.*
Assigned reading: [4, Section 2.1 and 2.3]
Define complex manifolds, complex submanifolds and analytic subvarieties, sheaf of holomorphic functions and holomorphic maps, field of meromorphic functions. Explain (do not prove) Siegel's theorem ([4, Proposition 2.1.9]). Give some examples, e.g. projective space, smooth (projective) hypersurfaces, grassmannian, complex tori and Hopf surfaces. Define divisors and line bundles and explain the link between the two ([4, Cor; 2.3.10]).

0.1.2. *Holomorphic vector bundles.*
Assigned reading: [4, Sections 2.2]
Define holomorphic vector bundles, insist on line bundles and the Picard group of a manifold. Discuss the tautological line bundle on the projective space ([4, Proposition 2.2.6]) and the canonical short exact sequences on the projective space and grassmannian. Discuss the exponential exact sequence and the first Chern class. Discuss the normal bundle exact sequence and the adjunction formula.

0.2. **Differential forms on complex manifolds (10.4.19).**

0.2.1. *Decomposition of the sheaves of differential forms.*
Assigned reading: [4, pp. 25-28], [4, Sections 1.3 and 2.6], [5, p. 75]
Define almost complex manifolds ([4, 2.6.2]). Prove [4, 2.6.4]. Explain [4, 2.6.8] and prove [4, 2.6.11]. Discuss the example of the form associated to a hermitian line bundle ([5, p. 75]). Discuss the link between almost complex and complex manifolds ([4, Prop. 2.6.15, 2.6.17, Thm 2.6.19]).

0.2.2. *Dolbeault complex and first Chern class.*
Assigned reading: [4, Section 2.6], [5, pp. 57-59, 162-164]
Define the Dolbeault complex of a holomorphic vector bundle and prove that it is a resolution of the vector bundle ([5, pp. 57-59, Prop. 4.19]). Define Dolbeault cohomology groups. Prove that the form associated to a hermitian line bundle represents the first Chern class of the line bundle ([5, pp. 162-164]).

0.3. **Kähler manifolds (17.4.19).**

0.3.1. *Definitions.*
Assigned reading: [4, pp. 28-31, pp. 48-49, pp. 116-120]
Present the infinitesimal (linear) picture of a euclidian structure compatible with a (almost) complex structure: introduce the fundamental form ω, prove [4, Lemmas 1.2.14, 1.2.15, 1.2.16, 1.2.17] and express ω in coordinates ([4, p.31]).
Explain the statement of [4, Proposition 1.3.12].
Define Kähler structures. Present [4, Lemma 3.1.7] and [4, Corollary 3.1.8].

0.3.2. *Examples.*
Assigned reading:[4, Section 3.1], [5, pp. 75-82]
Define the Fubini-Study metric [4, Examples 3.1.9]. Mention [4, Proposition 3.1.10 and Corollary 3.1.11]. Define the projectivization of a vector bundle and prove [5, Proposition 3.18]. Introduce the blow-up of a complex manifold along a complex submanifold and indicate the main ideas of [5, Proposition 3.24].

0.4. **Kähler identities (24.4.19).**

0.4.1. *Hermitian linear algebra.*
Assigned reading: [4, Section 1.2]
Define the Lefschetz operator [4, 1.2.18] and its dual [4, 1.2.21]. Prove that they define an $\mathfrak{sl}_2$-representation on $\bigwedge^* V^*$ [4, Proposition 1.2.26]. Prove the Lefschetz decomposition theorem [4, Proposition 1.2.30].

0.4.2. *Kähler algebra.*
Assigned reading: [4, Section 3.1]
Give an overview over the operators occurring in the Kähler identities [4, Proposition 3.1.12] and prove the identities.

0.5. **Hodge theory of Kähler manifolds (8.5.19).**

0.5.1. *Hodge decomposition.*
Assigned reading: [4, Section 3.2, 4.1]
Define the L^2-norm and the various spaces of harmonic forms and prove 3.2.6. State 3.2.8 and prove 3.2.9, 3.2.10 and 3.2.12. Use Remark 3.2.7 to deduce the corresponding symmetries for the cohomology groups. Present Serre duality ([4, Prop. 4.1.15]). Prove that Hopf surfaces cannot be endowed with a Kähler metric (Exercises 3.26 and 3.2.7).

0.5.2. *Hodge structures.*
Assigned reading: [4, p. 138], [5, Sections 7.1, 7.3]
Explain the diagram on [4, p. 138]. Introduce (integral) Hodge structures and their Hodge filtration. Prove that the category of (integral) Hodge structures is abelian ([5, 7.3.1]). Prove that the pullback by a holomorphic map between compact Kähler manifolds is a morphism of Hodge structures. Prove also [5, Lemma 7.28].

0.6. **Lefschetz decomposition (22.5.19).**

0.6.1. *Lefschetz theorems.*
Assigned reading: [4, Section 3.3]
Prove the Lefschetz $(1,1)$-theorem (3.3.1 and 3.3.2). Prove 3.3.10, define primitive cohomology and prove the Hard Lefschetz theorem (3.3.13). Prove the Hodge-Riemann bilinear relations (3.3.15).

0.6.2. *Hodge structures and Lefschetz theorems.*
Assigned reading: [5, pp. 178-180, 273-274, 284-287, 160]
Prove that the push-forward by a holomorphc map between compact Kähler manifolds is a morphism of Hodge structure ([5, pp. 178-180]). Prove (assuming [5, Thm. 11.21]) [5, Prop. 11.20] and state the Hodge conjecture. Define the tensor product of Hodge structure and prove [5, Lemma 11.41]. Define polarized Hodge structure and prove that the category of polarized Hodge structure is semi-simple ([5, Lemma 7.26]). Introduce the polarized Hodge structures associated to a projective manifold.

0.7. **Connections and curvature (29.5.19).**

0.7.1. *Connections.*
Assigned reading: [4, Section 4.2]
Introduce the Chern connection (4.2.14). Discuss Example 4.2.16 (ii). Explain the difference to a holomorphic connection and discuss (if time permits prove) 4.2.19.

0.7.2. *Curvature.*
Assigned reading: [4, Section 4.3], [5, pp. 228-231]
Define the curvature of a connection and discuss Lemmata 4.3.2 and 4.3.5 of [4]. Discuss [4, Prop. 4.3.8, 4.3.10]. Explain Example 4.3.12. Define local systems and the natural connection (Gauss-Manin) on the associated vector bundles ([5, pp. 228-229]). Prove [5, Prop. 9.11].

0.8. **Kodaira vanishing and embedding theorems (5.6.19).**

0.8.1. *Kodaira vanishing.*
Assigned reading: [4, Section 5.2]
Prove Kodaira vanishing theorem.

0.8.2. *Kodaira embedding.*
Assigned reading: [4, Section 5.3]
Explain how a complete linear system induces a closed embedding if and only if it separates points and tangent vectors. Prove the Kodaira embedding theorem.

0.9. **Deformation of complex manifolds (19.6.19).**

0.9.1. *First order deformations.*
Assigned reading:[5, Chapter 9]
Define deformation of complex manifolds. Explain the statements of Theorem 9.3 and Proposition 9.5. Define the Kodaira-Spencer map and prove [5, Prop. 9.7]. Prove the Cartan-Lie formula (Proposition 9.14).

0.9.2. *Local deformation.*
Assigned reading: [4, pp. 257-261], [5, Chapter 9]
Discuss local deformation ([4, Lemma 6.1.2, Corollary 6.1.6]). Explain [4, Prop. 6.1.11]. Mention the results of Proposition 9.20 and Theorem 9.23. Deformation of projective space, of projective hypersurfaces and of complex tori.

0.10. **Kähler non projective manifolds (26.6.19).**

0.10.1. *Complex tori and abelian varieties.*
Assigned reading: [6, Section 1 and 2]
Explain the equivalence of category between Hodge structure of weight 1 and complex tori (Example 1.8). Explain the consequences of Kodaira embedding (and Lefschetz theorems) for complex tori (1.5.2). Give an example of non projective complex torus (Proposition 2.8).

0.10.2. *Counter-example to the Kodaira problem.*
Assigned reading: [6, Section 1 and 2]
Introduce the Kodaira problem. Present (the main ideas of) Voisin's counter-example to the Kodaira problem.

0.11. **Variation of Hodge structure (3.7.19).**

0.11.1. *Period map.*
Assigned reading: [5, Chapter 10]
Define the period map of a (local) deformation of a compact Kähler manifold. Prove that the period map is holomorphic (Theorem 10.9) and Proposition 10.12.

0.11.2. *Properties and Torelli theorems.*
Assigned reading: [5, Chapter 10]
Define Hodge bundles and prove transversality theorem (Proposition 10.18). Prove Lemma 10.19 and Theorem 10.21.

0.12. **Last session (10.7.19).**

0.12.1. *Torelli theorems.*
Assigned reading: [5, Chapter 10]
Prove Lemma 10.22 and Corollaries 10.25 and 10.26. Prove Theorem 10.27.

0.12.2. *Griffiths' intermediate Jacobian.*
Assigned reading: [5, Chapter 12]
Define the intermediate Jacobian associated to a Hodge structure of odd weight and construct the Abel-Jacobi map. Explain Theorem 12.21.

References

[1] M. A. de Cataldo, *The hodge theory of projective manifolds*, Imperial College Press, 2007.
[2] J.-P. Demailly, *Complex analytic and differential geometry*, notes.
[3] P. Griffiths, J. Harris, *Principles of algebraic geometry*, John Wiley & sons, New York, 1978.
[4] D. Huybrechts, *Complex geometry*, Springer, Berlin, 2005.
[5] C. Voisin, *Hodge theory and complex algebraic geometry I*, Cambridge University Press, Cambridge, 2002.
[6] C. Voisin, *Algebraic geometry versus Kähler geometry*, Milan J. Math., 78(1): 85-116, 2010 notes