Graduate Seminar on Algebra (Modul S4A3)
Hodge theory of complex manifolds, Winter term 2013/14

Time and location: Thursday 2:15-3:45 pm, SR 0.008, Begin October 17

Prerequisites: Basic knowledge of complex manifolds, e.g. Hodge decomposition for Kähler manifolds. The seminar is suitable for participants of last term's seminar 'Complex geometry' and for everyone familiar with the main topics of [5, Ch. 1, 2] or [1, 7]. The main reference for this term is [10], but you should free to use also sources that are not listed below.

Talks: If you are interested in giving a talk or are at least generally interested in participating, please contact me via email *as soon as possible.* I will try to distribute at least the first talks before the start of the term.

Below a sketch of the first few talks, but this is flexible. Some of the topics might need two talks. It would be good to form teams and maybe split talks.

1. Hodge–Frölicher spectral sequence. A. Mihatsch (?)
– Recall the Hodge decomposition of a compact Kähler manifold.
– Discuss the holomorphic de Rham complex $\Omega_X^\bullet$, see [10, 8.2].
– Introduce the Hodge filtration $F^p\Omega_X^\bullet$, see [10, 8.3.3]. See also the discussion in [10, 12.3.1], but Deligne cohomology will be introduced only later.
– Recall the necessary facts from homological algebra that a filtration of a complex leads to a spectral sequence. This is all recalled in [10, 8.3.1 and 8.3.2] but you may consult any of the standard sources (Weibel, Gelfand–Manin, etc.).
– Deduce the Hodge–Frölicher spectral sequence

$$E_1^{p,q} = H^{p,q}(X) \Rightarrow H^n(X, \mathbf{C}).$$

– Conclude that it degenerates for compact Kähler manifolds. (It always degenerates for compact complex surfaces, see [2]. See also [7, Exercise 3.2.4].)

2. Albanese. J. Witaszek
– Recall the construction of the Picard variety $\mathrm{Pic}^0(X)$ and then introduce the Albanese variety $\mathrm{Alb}(X)$ and the Albanese map $X \to \mathrm{Alb}(X)$, see [7, 3.3].
– Show that $\mathrm{Pic}^0(X)$ and $\mathrm{Alb}(X)$ are dual to each other.
– Prove that the Abel–Jacobi map $\mathrm{alb} : X \to \mathrm{Alb}(X)$ is holomorphic. Generalize this to $\mathrm{alb}^k : X^k \to \mathrm{Alb}(X)$ and prove the surjectivity for $k \gg 0$, see [10, Lemma 12.11].
– Universal property of the Albanese map [7, Prop. 3.3.8].
– Analyze Picard and Albanese variety in the case of a complex torus.
– Use the Hodge–Riemann pairing to prove that $\mathrm{Pic}^0(X)$ and $\mathrm{Alb}(X)$ are abelian varieties for projective X.
– Discuss the two notions for curves.

3. Intermediate Jacobian. M. Schmidt
– Introduce the intermediate Jacobian

$$J^{2k-1}(X) = H^{2k-1}(X, \mathbf{C})/(F^k H^{2k-1}(X) \oplus H^{2k-1}(X, \mathbf{Z}))$$

as a generalization of the Albanese and Picard variety. See [4, Sect. 6] and [10, Ch. 12].
– Mention that the intermediate Jacobian need not be projective (even for projective X). Define the Abel–Jacobi map (without proof of holomorphicity, which is analogous to the Albanese map). See [4, pp. 27,28]. Explain why it factors through the Chow group (or

postpone to corresponding statement for Deligne cohomology, see below).
– Mention that the Abel–Jacobi map for the Picard variety is just c_1 (see Prop 12.7 or [7], no proof).
– Compute the infinitesimal Abel–Jacobi map (see [4, Remark p. 28] and [10, Lemma 12.6]).

4. Deligne cohomology. E. Reinecke
– Introduce the Deligne complex, see [10, 12.3.1] and prove the exact sequence

$$0 \to J^{2k-1}(X) \to H_D^{2k}(X, \mathbf{Z}(k)) \to \mathrm{Hdg}^{2k}(X, \mathbf{Z}) \to 0.$$

You may want to recall the Hodge conjecture at this point. (Note that there exists an algebraic construction of the Albanese variety, say for varieties over algebraically closed fields, but none for the intermediate Jacobian, simply because it is not projective in general.)
– Discuss low degree cases in [3, Sect. 1.4, 15, Lemma 1.6]. Define the cup-product for Deligne cohomology classes.
– Sketch the construction of fundamental classes in Deligne cohomology, see [3, Sect. 7.1], [9, Sect. 7.2] or the long explanation in [10, 12.3.2, 12.3.3]). Explain why it factors through Chow groups [3, Prop. 7.6, Cor. 7.7]. Compare it to the Abel–Jacobi map. View the intermediate Jacobians as an ideal of square zero (see [10, Prop. 7.10]).
– Include a short discussion of Chern classes in Deligne cohomology via splitting principle, see [3, Sect. 8] and maybe the recent [6].

5. Families of complex manifolds. Ch. Hemminghaus, B. Öner
– Introduce the notion of a family and state Ehresmann's theorem, see [10, Thm. 9.3].
– Discuss the local systems $R^if_*\mathbf{R}$, $R^if_*\mathbf{Q}$, etc.
– Introduce the Kodaira–Spencer map. See [10, 9.1.2], but it might be preferable to introduce it simply as the boundary map of the associated long cohomology sequence and simply state the description of its Dolbeault representative. Mention the version over $\mathrm{Spec} k[\varepsilon]$.
– Recall the notion of connection and curvature. Include a brief discussion of the relation between locally constant systems and flat connections. State the result and give an idea, but no proof.
– Introduce the Gauss–Manin connection, see [10, 9.2].

6. Variation of Hodge structures. TBC
– Recall the abstract notion of a (pure) Hodge structure and its geometric realization by the cohomology of a compact Kähler manifold.
– Define the period map and prove that it is holomorphic, see [10, 10.1.2]. It could easier and more instructive to follow the discussion in the special case of K3 surfaces in [8, Ch. 6.2] and then just state the results in the general case.
– State and prove Griffiths transversality [4, Thm. p. 32]. See also [10, Ch. 17.1].
– Introduce the abstract notion of a variation of Hodge structures (VHS).

7. Lefschetz pencil. TBC
– Recall the Lefschetz hyperplane theorem, see [7, Prop. 5.2.6], and state its integral version [10, Thm. 13.1].
– Define ordinary double points of hypersurfaces and the notion of a Lefschetz pencil, see [10, Sec. 14.1].
– State and prove the existence of Lefschetz pencils.
– Vanishing spheres, [10, Sec. 14.2.1].
– Primitive and vanishing cohomology. Also recall the notion as introduced in [7, Ch. 3.3].

8. Monodromy.
– Define the monodromy action.

– Prove the Picard–Lefschetz formula, [10, Prop. 15.3].
– Prove the irreducibility of the monodromy representation, [10, Thm. 15.4].

9. Deligne's theorem on invariant cycles. [10, Ch. 16.3]
– State the existence of the Lefschetz spectral sequence (we assume everybody knows this).
– Prove its degeneration (proved by Deligne at the age of 24). One could follow e.g. [10, Thm. 16.15], but there other accounts of it.
– Explain the monodromy representation and prove [10, Thm. 16.24]. This is the result for smooth morphisms!

List of talks.

17.10.: NN

24.10.: NN

31.10.: No seminar.

7.11.: NN

14.11.: NN

21.11.: NN

28.11.: NN

5.12.: NN

12.12.: NN

19.12.: Possibly no seminar.

References

[1] W. Ballmann *Lectures on Kähler manifolds*, ESI Lectures in Mathematics and Physics, EMS, 2006.

[2] W. Barth, K. Hulek, C. Peters, A. van den Ven *Compact complex surfaces*, Springer

[3] H. Esnault, E. Viehweg *Deligne–Beilinson cohomology*, in *Beilinson's conjectures on special values of L-functions*, ed. Rapoport, Schappacher, Schneider. Perspectives in Math. Academic Press (1988).

[4] M. Green *Infinitesimal Methods in Hodge theory*, CIME Notes. Springer 1994

[5] P. Griffiths, J. Harris *Principles of algebraic geometry*, Wiley 1978.

[6] J. Grivaux *Chern classes in Deligne cohomology for coherent analytic sheaves*, Math. Ann. 347 (2010), no. 2, 249–284. arXiv:0712.2207.

[7] D. Huybrechts *Complex geometry*, Springer Universitext (2004).

[8] D. Huybrechts *Lectures on K3 surfaces*, `http://www.math.uni-bonn.de/people/huybrech/K3.html`

[9] C. Peters, J. Steenbrink *Mixed Hodge Structures*, Springer (2008).

[10] C. Voisin *Théorie de Hodge et géométrie algébrique complexe*, Cours spécialisés 10. SMF (2002). (or the english translation, published by Cambridge)