Graduate Seminar on Algebra (Modul S4A1)
Advanced topics in Hodge theory

This is a first attempt. For some of the talks I have sketched a complete plan, for others only the main points. Some of the talks may take longer than just one session. Also, the order of some of the talks can be changed, e.g. V and VI could come before III and IV. The focus is on complex algebraic techniques, so we will only sketch or survey analytical and topological techniques that come in at points.

I. Intermediate Jacobians (Ref: [3, Sect. 6], [9, Ch. 12])

1. Recall (briefly): Hodge decomposition, harmonic forms, Fröhlicher spectral sequence. Introduce the notation F^pH^q, recall the definition of a pure Hodge structure. Blend in the proposition on p.5 in [7]. Discuss the short exact sequence before Cor. 12.27 in [9]

2. Albanese $Alb(X)$: Explain the construction, mention that it is dual to the Picard variety, and introduce the polarization for X projective. Prove that the Abel–Jacobi map $alb : X \to Alb(X)$ is holomorphic. Generalize this to $alb^k : X^k \to Alb(X)$ and prove the surjectivity for $k \gg 0$ ([9, Lemma 12.11]. Prove projectivity once more via the Moishezon criterion ([9, Cor 12.12]). Explain the universality property of the Albanese. Construction in the algebraic category (Serre, see reference in [9], also Igusa?).

3. Intermediate Jacobians. Explain the construction. Mention that the intermediate Jacobian need not be projective (even for projective X). Define the Abel–Jacobi map (without proof of holomorphicity, which is analogous to the Albanese map). See [3, pp. 27,28]. Explain why it factors through the Chow group (or postpone to corresponding statement for Deligne cohomology, see below). Compute the infinitesimal Abel–Jacobi map (see [3, Remark p. 28] and [9, Lemma 12.6]). View Albanese and Picard variety as special cases. Mention that the Abel–Jacobi map for the Picard variety is just c_1 (see Prop 12.7 or [5], no proof).

4. Define the algebraic part of the intermediate Jacobian. (Check the literature whether there is an algebraic construction.) Consider homological modulo algebraic equivalence and define the Griffiths group. State Griffiths' result for the quintic threefold saying that there are non-torsion points in the Griffiths group. Mention the results of Clemens and Voisin that the Griffiths group is not necessarily finitely generated [9, Thm. 12.21]. State the conjecture asserting the surjectivity of the Abel–Jacobi map (onto the algebraic part) and show that it would follow from the Hodge conjecture. Prove it for uniruled threefolds, see [9, Exercise Ch. 12].

5. Intermediate Jacobian in families. Beginning of Lecture 6 in [3].

II. Deligne cohomology (Ref: [9], [2], [3], [6])

1. Define Deligne cohomology [9, Sect. 12.3] (we only deal with the absolute case for the moment). Explain the elation to Hodge cycles and to the intermediate Jacobian [9, Prop. 12.26]. Discuss low degree cases in [2, Sect. 1.4, 15, Lemma 1.6]). Define the cup-product for Deligne cohomology classes.

2. Sketch the construction of fundamental classes in Deligne cohomology, see [2, Sect. 7.1], [6, Sect. 7.2] or the long explanation in [9, 12.3.2, 12.3.3]). Explain why it factors through Chow groups [2, Prop. 7.6, Cor. 7.7]. Compare it to the Abel–Jacobi map. View the intermediate Jacobians as an ideal of square zero (see [9, Prop. 7.10]).

3. Include a short discussion of Chern classes in Deligne cohomology via splitting principle. ([2, Sect. 8] and the recent [4]).

III. Variation of Hodge structures (Ref: [3, Lect. 3], [9, Ch. 17.1])

1. Define the Kodaira–Spencer map [3, p. 31], [9, Sect. 9.1.29]. Mention the version over Spec$k[\varepsilon]$.

2. Explain the Gauss–Manin connection [9, Ch. 9.2]. Combine with V.1 (see the comments there). State and prove Griffiths transversality [3, Thm. p. 32]. Introduce period domain and period map [9, Ch. 10.1], [3, p. 31,32] (see references in [3]). (Maybe Yukawa, Wahl)

3. We could discuss the infinitesimal Torelli theorem for hypersurfaces here [3, Lect. 4], but maybe better later?

IV. Hodge locus [9, Ch. 17.3]

1. Define the Hodge locus [9, Def. 12.12] and derive the local description [9, Prop. 17.14].

2. Add the first order description of the Hodge locus [9, Lemma 17.16].

3. The components of the Noether–Lefschetz are the Hodge loci for degree two classes. Discuss the case of curves in surfaces [9, Prop. 17.19]. In interesting cases (e.g. for K3 surfaces and hyperkähler manifolds) the Noether–Lefschetz locus is dense [9, Prop. 17.20].

3. We could (give an out)look at the more recent [1] who prove that the Hodge locus is algebraic. This is predicted by the Hodge conjecture. This could be sketched.

V. Monodromy action, Lefschetz pencils and vanishing cycles (Ref: [9, Ch. 14, 15]

1. Assume or just state equivalence between locally constant systems and monodromy reps. Explain how they occur for families.

2. State that the monodromy operator T is quasi-unipotent [9, Thm. 15.15] (we won't prove this).

3. Introduce Lefschetz pencils and the Picard–Lefschetz operators ([9, Thm. 15.16]). We need to cut short the topology, more details later.

4. Irreducibility of the monodromy action on the space of vanishing cycles [9, Thm. 15.27]). Application to Noether–Lefschetz [9, Sect. 15.3].

VI. Deligne's theorem on invariant cycles [9, Ch. 16.3]

1. State the existence of the Lefschetz spectral sequence (we assume everybody knows this).

2. Prove its degeneration (due to Deligne). One could follow e.g. [9, Thm. 16.15]. Have a look at the more categorical approach in [8].

3. Explain the monodromy representation and prove [9, Thm. 16.18]. This is the result for smooth morphisms!

VII. Mixed Hodge structures [9, Ch. 16.3.2] and invariant cycles

1. Recall the abstract notion of a mixed Hodge structure and state and prove Deligne's theorem that morphisms of MHS are strict (see [9, Thm. 16.12])

2. Prove Deligne's theorem on invariant cycles for the case of non-smooth morphisms [9, Thm. 16.24]. State the formal properties of the MHS that are needed.

3. Construction of MHS on quasi-projective varieties: We need the logarithmic de Rham complex etc. (see [9, Ch. 8.4] or [3, Lect. 5]). This part is long and we might want to postpone it.

From here, we should feel free to change the program. We could choose from the following list.

Further topics: VIII. Generic Torelli for Hypersurfaces [9, Ch. 18], [3, Lect. 4, 7] IX. Normal functions [3, Lect. 6], [9, Ch. 19], X. Nori connectedness [3, Lect. 8], [9, Ch. 20], XI. Chow, Mumford, Bloch [9, Ch. 21-23]

Organization: Let us try something new: We assign a 'leader' to the various parts of the seminar. The leader does not necessarily have to (or maybe should not) give the talks, but he should feel particularly responsible for their success. He should know the part as well as the speaker and should talk to him before. Both should talk to me anyway if necessary.

Here is my suggestion for the first three bits. Since we might change the program, more does not make much sense at this point. Feel free to object.

I, II: Uli Schlickewei

III, IV: Sven Meinhardt

V, VI: Heinrich Hartmann

References

[1] E. Cattani, P. Deligne, A. Kaplan *On the locus of Hodge classes*, JAMS. 8, (1995), 483–506.

[2] H. Esnault, E. Viehweg *Deligne–Beilinson cohomology*, in [7].

[3] M. Green *Infinitesimal Methods in Hodge theory*, CIME Notes. Springer 1994

[4] J. Grivaux *Chern classes in Deligne cohomology for coherent analytic sheaves*, arXiv:0712.2207.

[5] D. Huybrechts *Complex geometry*, Springer Universitext (2004).

[6] C. Peters, J. Steenbrink *Mixed Hodge Structures*, Springer (2008).

[7] M. Rapoport, N. Schappacher, P. Schneider *Beilinson's conjectures on special values of L-functions*, Perspectives in Math. Academic Press (1988).

[8] M. Van den Bergh *A remark on a theorem by Deligne*, arXiv:math/0212197.

[9] C. Voisin *Théorie de Hodge et géométrie algébrique complexe*, Cours spécialisés 10. SMF (2002). (or the english translation, published by Cambridge)