Graduate Seminar on Algebra (Modul S4A1)
Advanced topics in Hodge theory and Chow groups of algebraic varieties

This seminar is a continuation of last terms seminar but technically rather independent of it. We will follow [V] and work through Chapter 21-23. (This corresponds to III.9-11 in Vol 2 of the English translation.) We will start with the definition of Chow groups and their basic properties. For more details we use [F]. The aim of the seminar is to understand Mumford's theorem saying that under certain cohomological conditions on the variety, the Chow group is of infinite dimension. The last part treats geometric aspects of the Bloch conjecture which predicts the existence of a natural filtration of the Chow ring whose graded pieces are governed by the more transcendental parts of cohomology. The seminar should be accessible to everyone with a sound knowledge of algebraic geometry. Later in the seminar we will use the Hodge theoretic aspects introduced last term (which one might accept as a black box). If you are interested in participating, please contact me (huybrech@) or Sönke Rollenske (srollens@) directly or come to the first meeting Friday April 17. If you are interested in giving a talk, please contact Pawel Sosna (sosna@) for Part I, Sönke Rollenske (srollens@) for Part II or Heinrich Hartmann (hartmann@) for Part III.

Literatur

[F] W. Fulton Intersection theory. Springer.

[V] C. Voisin, Théorie de Hodge et géométrie algébrique complexe (or the English translation published by Cambridge)

Plan

Some of the talks will take longer than 90 min. Some of the potentially more time consuming bits are marked with $*$. Other talks might be shorter, especially if the participants know already about the basic properties of Chow groups.

I. Chow groups

This part recalls the construction of the Chow group and its ring structure. In general Chow groups are difficult to compute in geometric interesting examples. We will work out a few example in detail. Eventually, Chow groups are governed at least conjecturally by the Hodge theory of the variety. So we will need to make the connection to the cohomology via cycle and Abel–Jacobi map.

I.1. Based on Section 21.1. in [V]. Recall the definition of $\mathrm{CH}(X)$. At this point we should work over an arbitrary base field and specialize to $\mathbf{C}$ only later when we apply Hodge theory. Topics of this first talk should be: i) The direct image $p_* : \mathrm{CH}(Y) \to \mathrm{CH}(X)$ for proper morphisms $p : Y \to X$. ii) The pull-back $p^* : \mathrm{CH}(X) \to \mathrm{CH}(Y)$ for flat morphisms p (later for arbitrary ones). iii) The localization sequence (Lemma 21.12.). iv) Homotopy invariance $\mathrm{CH}(X) = \mathrm{CH}(X \times \mathbf{A}^1)$ (Prop. 21.11.). v) Chow groups of vector bundles (Thm. 21.13.).

I.2. Based on Section 21.2. in [V], but use [F] for more details. i) Use the normal cone construction in [F] for the intersection product. ii) Projection formula and compatibility with pull-back (Prop. 21.14.) iii) Correspondences and their composition (Prop. 21.17.) We could hint at the derived version for Fourier–Mukai transforms. iv) Chow moving lemma (Lemma 21.22.) v) Recall the notion of the cycle classes (in cohomology, Deligne cohomology, and/or intermediate Jacobians). v) Discuss the multiplicative property of the Abel–Jacobi map

(Prop. 21.23.) vi) Work out the examples 1-3 in Section 21.3. (curves, projective bundles, blow-ups). Hint at the derived versions of these statements.

I.3. Based on Section 6.3. in [F]. Excess intersection formula. The prototype of an excess intersection formula is $i^*i_*\alpha = c_d(\mathcal{N}_{X/Y}) \cap \alpha$ for α a class on a smooth subvariety $i : X \hookrightarrow Y$ of codimension d. For simplicity we may restrict to closed smooth embeddings. Compare this with the formula $\mathcal{H}^\ell(Li^*i_*F^\bullet) = \bigoplus_{s-r=\ell} \bigwedge^r N^*_{X/Y} \otimes \mathcal{H}^s(F^\bullet)$ for a complex of coherent sheaves $F^\bullet$ on Y. An elementary introduction to excess intersection formula can be found in Katz's book 'Enumerative geometry and string theory'. For excess intersection for divisor one could use Lazarsfeld's article in Compositio 43 (1981).

I.4. Based on Section 21.2.4. in [V]. i) Chow groups of hypersurfaces of small degree. ii) Work out in detail the two examples in the exercises: Chow groups of conic bundles and the decomposition of the diagonal of a surface (Murre's theorem).

II. Mumford's theorem

The Chow group of zero-dimensional cycles is studied in detail. We will see examples where this group is essentially trivial and others where it is quite big.

II.1. Based on Section 22.1. in [V]. Chow groups can be enormous, but it is a priori not clear how to measure their size. i) Define the natural maps $\sigma : X^{(d)} \times X^{(d)} \to \mathrm{CH}_0(X)$ and say when $\mathrm{CH}_0(X)$ is representable. ii) Define when $\mathrm{CH}_0(X)$ is called finite dimensional and prove Prop. 22.10. saying that both definitions coincide. iii)* Prove Roitman's theorem that for finite dimensional Chow groups the Abel–Jacobi map identifies $\mathrm{CH}_0(X)_{\mathrm{hom}}$ and $\mathrm{Alb}(X)$. (Thm. 22.11.)

II.2. The proof of Roitman's theorem 22.11. relies on the more famous one showing that $\mathrm{alb} : \mathrm{CH}_0(X)_{\mathrm{hom}} \to \mathrm{Alb}(X)$ is an isomorphism on the torsion points. This theorem is not proved in [V]. We should try to follow Bloch's version of it (Compositio 39, 1979).

II.3. Based on Section 22.1.3. and 22.2. in [V]. i) State the three Theorems 22.15., 22.16., 22.17. Mumford's theorem 22.15. shows that under a certain assumption on the Hodge theory the Chow group cannot be finite dimensional. Find examples of surfaces where the assumption is satisfied. Corollary 22.18. is a trivial consequence. ii) Deduce Thm. 22.16. from Thm. 22.17. iii) Apparently Thm. 22.17. also implies Roitman's theorem of the last talk. Can you explain why? iv)* Explain and prove the Bloch–Srinivas decomposition. Theorem 22.19. v) Deduce from it Mumford's theorem (Section 22.2.2)

II.4. Based on Section 22.2.3. in [V]. i) Prove Prop. 22.26. saying that if $\mathrm{CH}_0(X)$ has the size of $\mathrm{CH}_0(X')$ of a three dimensional subvariety $X' \subset X$, then the Hodge conjecture holds for X. ii) Prove Prop. 22.27. which shows that the Griffiths group is torsion if $\dim(X') \geq 2$. (Recall the notion of the Griffiths group and what we have learned about it last term).

II.5. Based on Section 22.3. in [V]. What can we conclude when the cycle map is injective? i) Prove Thm. 22.29. which then decomposes the diagonal. ii) Prove Thm. 22.31. which then deduces information about the Hodge structure. iii) Work out Exercise 2 (Standard conjecture on the algebraicity of the dual Lefschetz operator).

III. Bloch conjecture

There are many conjectures of Bloch, most of them are related to the one that shall be discussed here. It conjectures that $\mathrm{alb} : \mathrm{CH}_0(X)_{\mathrm{hom}} \to \mathrm{Alb}(X)$ is an isomorphism for any smooth complex projective surface with $H^{2,0}(X) = 0$. (Recall that Mumford's theorem is concerned with surfaces with $H^{2,0}(X) \neq 0$ for which the injectivity definitely does not hold.)

The conjecture has been established for special surfaces, e.g. Enriques surfaces, but is wide open in general. The conjecture fits in a more general framework concerning certain natural filtration of the Chow group whose graded parts are governed by the various bits of the Hodge theory of the variety.

III.1. Based on Section 23.1. in [V]. The conjecture should be discussed for specific types of surfaces. E.g. it can be proved for all surfaces which are not of general type. Some parts of the classification theory of surfaces is needed for this. We will decide later in what detail this should be presented.

III.2. Based on Section 23.2. in [V]. i) Discuss the conjecture filtration of the Chow group and its expected properties. (Conj. 23.21.) ii) Show that Conjecture 23.22. implies 23.23., which states injectivity of the cycle map under certain conditions on the Hodge structure. iii) There are various candidates for the filtration. Present Saito's one for which finiteness has not yet been shown (Section 23.2.3.).

III.3. The Bloch filtration has been studied in greater detail for abelian varieties by Bloch and Beauville (see Section 23.3.). For symplectic variety the conjectured filtration is expected to split. For K3 surfaces, this has been proved for K3 surfaces by Beauville and Voisin. There are partial results and conjectures in higher dimensions in particular for Hilbert schemes of points on K3 surfaces (again due to Beauville and Voisin). We will decide later which of these examples shall be discussed (if time permits at all).