Spin-Structures on Riemannian Manifolds and the Spinorbundle

Spin structures of a principle $SO(n)$-bundle

Let X be a connected CW complex, e.g. a connected Riemannian manifold, and let a principle $SO(n)$-bundle be denoted with $(Q, \pi, X; SO(n))$.

Definition 1 *A **Spin-structure of** Q is a pair (P, Λ) where*

- *P is a principle $Spin(n)$-bundle and*
- *Λ is a 2-sheeted covering that preserves the group action i.e. the following diagram commutes*

$$\begin{array}{ccccc} PxSpin(n) & \longrightarrow & P & & \\ & & & \searrow^{\pi} & \\ \downarrow \Lambda x\lambda & & \downarrow \lambda & & X \\ & & & \nearrow_{\pi} & \\ QxSO(n) & \longrightarrow & Q & & \end{array}$$

Let λ be the covering of $SO(n)$by $Spin(n)$.

Recall that the fiber F of the bundle Q is isomorphic to $SO(n) \cong F$, thus for $n \geq 3$ $\Pi_1(F) = \mathbb{Z}_2$ since $SO(n)$ has two connected components. For a given Spin-structure (P, Λ) of Q regard the subgroup $H(P, \Lambda) := \Lambda_*(\Pi_1(P)) \subset \Pi_1(Q)$ generated by the image of the 2-sheeted covering. Therefore $H(P, \Lambda)$ is of index 2.

Theorem 1 *Let $\alpha_F := \iota_\#(\alpha) \in \Pi1(Q)$ where α is the nontrivial element in $\Pi_1(F) = \mathbb{Z}_2$ and ι_* the induced map of the embedding $\iota : F \longrightarrow Q$, then:*
$\alpha_F \notin H(P, \Lambda)$.

Definition 2 *Two Spin-structures (P_1, Λ_1) and (P_2, Λ_2) are called **equivalent** if a Spin - equivariant map exists such that $\Lambda_1 = \Lambda_2 \circ f : P_1 \rightarrow P_2 \rightarrow Q$.*

Example 1 *Two Spin-structures of Q don't have to be equivalent even if the the corresponding principle Spin(n)-bundles are.*
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Theorem 2 *There is a bijective map from the set of equivalence-classes of Spin-structures for Q and the subgroups $H \subset \Pi_1(Q)$ of index 2 such that $\alpha_F \notin H(P, \Lambda)$.*

Corollary 1 *The Spin-structures of Q (principle $SO(n)$ bundle over a connected CW complex X) are in one to one correspondence to all homomorphisms $f : \Pi_1(Q) \rightarrow \Pi_1(F)$ with $f \circ \iota_\# = id_{\Pi_1(F)}$.*

We get another Cor. regarding the exact sequence

$$\cdots \longrightarrow \Pi_2(X) \xrightarrow{\delta} \Pi_1(F) \underset{f}{\overset{\iota_\#}{\rightleftarrows}} \Pi_1(Q) \xrightarrow{\pi_\#} \Pi_1(X)$$

Corollary 2 *If a principle $SO(n)$bundle over a connected mfld X has a Spin-structure then (since $f \circ \iota_{\#} = id_{\Pi_1(F)}$)*

- $\Pi_1(Q) = \Pi_1(F) \oplus \Pi_1(X)$
- $\Pi_2(Q) = \Pi_2(X)$

And with the given corollaries we can see

Theorem 3 *Let X be a simply connected CW complex. A principle $SO(n)$bundle has a Spin-structure if and only if $\Pi_1(Q) = \mathbb{Z}_2$*

Theorem 4 *Let $1 \in \mathbb{Z}_2 = H^1(F, \mathbb{Z}_2)$ be the nontrivial element, then denote by $\delta(1) \in H^2(X, \mathbb{Z}_2)$, where δ is the coboundary map, the **2nd Stiefel-Whitney Class** of the principle $SO(n)$bundle. $\delta(1) =: \omega_2(Q)$.*
Then does the principle $SO(n)$-bundle have a Spin-structure if and only if $\omega_2(Q) = 0$. The set of all Spin-structures is classified by $H^1(X; \mathbb{Z}_2)$.

For the proof of this more general Thm (if X is a riemannian mfld) one has to remark, that we can describe the principle $SO(n)$bundle Q in terms of locally trivial fibrations with typical fiber $SO(n)$ (which is a Liegroup) and transition functions satisfying the cocycle conditions. $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$ and $g_{\alpha\beta} = e$ where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G = SO(n)$ and U_α is a fibre bundle atlas.

Definition 3 *A **Spin-structure of** Q is a lift of the transition functions to Spin(n) preserving the cocycle conditions, i.e. let g' denote the lifted functions then $\lambda(g'_{\alpha\beta}) = g_{\alpha\beta}$, $g'_{\alpha\beta} g'_{\beta\gamma} g'_{\gamma\alpha} = e$ and $g'_{\alpha\alpha} = e$ where $U_{\alpha\alpha}$is a cover of X such that Q is a trivial fibre bundle over U_α.*

If we denote with $\overrightarrow{s}_\alpha$ the local oriented orthonormal frames over U_α, the $g_{\alpha\beta}$ help to express the $\overrightarrow{s}_\beta$ (orthonormal frames over U_β): $\overrightarrow{s}_\alpha = g_{\alpha\beta} \overrightarrow{s}_\beta$.

Definition 4
- *A **Čech j-cochain** is a totally symmetric function $f(\alpha_0, ..., \alpha_j) \in \mathbb{Z}$ defined for indices $\alpha_0, ..., \alpha_j$ such that $U_{\alpha_0} \cap ... \cap U_{\alpha_j} \neq \emptyset$ i.e $f(\sigma(\alpha_0), ..., \sigma(\alpha_j)) = f(\alpha_0, ..., \alpha_j)$ for all permutations σ.*
- *Denote with $C^j((X, \mathbb{Z}_2)$ the **multiplicative group of all Čech j-cochains**.*
  And let $\delta_j : C^j(X, \mathbb{Z}_2) \to C^{j+1}(X, \mathbb{Z}_2)$ be the coboundary defined by

  $$(\delta f)(\alpha_0, ..., \alpha_{j+1}) := \prod_{i=0}^{j+1} f(\alpha_0, ..., \check{\alpha}_i, ..., \alpha_{j+1})$$

- *Denote by $H^j(X, \mathbb{Z}_2) := ker(\delta_j)/im(\delta_{j+1})$ the **j-th Čech Cohomology group**.*

Remark 1 *The multiplicative 1 is the 1-fct and $\delta^2 f = 1$ can easily be computed.*
The j-th Čech Cohomology group is independent of the choice of the covering.

Example 2 *Regard the complex projective space $\mathbb{C}P^n$. It is known that $SO(n)$operates naturally on $\mathbb{C}^{n+1}$ and with this on $\mathbb{C}P^n$ where we have to regard the group under which a point of $\mathbb{C}P^n$ is invariant, i.e. $\tilde{A} \in SO(n+1) : \tilde{A} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix}$, $\lambda \neq 0$. With this we have the* *isotropygroup given by* $\tilde{A} = \begin{pmatrix} & & & 0 \\ A & & & . \\ & & & . \\ 0 & . & . & a \end{pmatrix}$ *where* $a = 1/det(A)$ *and* $A \in SO(n)$.

Now let us regard a representation of this group $\sigma : S(U(n) \times U(1)) \to \underbrace{U(n)}_{\subset Aut(\mathbb{C}^n)} \subset SO(2n)$,

$\sigma(\tilde{A} = A * 1/det(A) \in U(n)$.
Let $R = SU(n+1) \times_\sigma SO(2n)$ *be the frame bundle. R is a principle* $SO(n)$*bundle.*

$$\begin{array}{ccc} R & \xleftarrow[Wirkg.]{\qquad} & G = SO(2n) \\ \downarrow \pi & & \\ \mathbb{C}P^n & & \end{array}$$

The Spin-structures of R are classified by the fundamental group. Since $\mathbb{C}P^n$ *is simple connected and* $\Pi_1(R)$ *is a surjective image of* $Pi_1(SO(n)) = \mathbb{Z}_2$ *the group* $\Pi_1(R)$ *has at most 2 elements. The representation map* σ *induces a homomorphism* $\sigma_\# : \Pi_1(S(U(n) \times U(1)) = \mathbb{Z} \to \Pi_1(SO(2n)) = \mathbb{Z}_2$, $\sigma_\#(1) = (n+1)mod2$. *One now can compute* $\Pi_1(R) = \begin{cases} \mathbb{Z}_2 & n = 2k+1 \\ 1 & n = 2k \end{cases}$ *such that only in the odd case a Spin-structure is given.*

Associated Spinorbundle

Definition 5 *Let*$(P, \pi, X; G)$ *be a principle G bundle . Regard the homeomorphisms Homeo(F) together with the compact open topology, F a topological space* $\forall \rho : G \to Homeo(F)$ *construct a fiber bundle over X with fiber F with a left action of G on* $P \times F \ni (p, f)$ *i.e.* $\phi_g(p, f) = (pg^{-1}, \rho(g)f), g \in G$ *and* $P \times_\rho F := P \times F/ \sim$ *where* $(p, f) \sim (q, h)$ *if* $\exists g \in Spin(n)$ *with* $\phi_g(p, f) = (q, h)$. *Then one derives a mapping* $P \times F \to P \to X$ *by projection. Therefore we have* $\pi_\rho : P \times_\rho F \to X$. *This construction is called* ***the associated bundle to P by*** ρ.

Now we regard a principle $SO(n)$bundle Q, the associated real n-dimensional bundle $T = Q \times_{SO(n)} \mathbb{R}^n$ and a Spin-structure (P, Λ).

Definition 6 *The representation* $\kappa : Spin(n) \to U(\triangle_n)$, $\triangle_n := \mathbb{C}^{2^k}$ *for* $n = 2k+1, n = 2k$ *allows us to regard:* $S := P \times_\kappa \triangle_n = P \times \triangle_n / \sim$, $(p, \delta) \sim (p', \delta')$ *if* $\exists g \in Spin(n)$ *with* $(p, g^{-1}, \kappa(g)\delta) = (p', \delta')$ *called the* ***Spinorbundle*** *od P.*

Lemma 1 *If* $n = 2k$ *then* $\triangle_n$ *splits:* $\triangle_n = \triangle_n^+ \oplus \triangle_n^-$.

Lemma 2 *The Clifford multiplication* $\mu : \mathbb{R}^n \otimes_\mathbb{R} \triangle_n \to \triangle_n$, $(\mu : \Lambda(\mathbb{R}^n) \otimes_\mathbb{R} \triangle_n \to \triangle_n)$ *is a homomorphism of the Spin-representation, i.e.* $\kappa(g)(\mu(x, \psi) = \mu(\lambda(g)x, \kappa(g)\psi)$, $(\kappa(g)(\mu(\omega^k, \psi)) = \mu(\lambda(g)\omega^k, \kappa(g)\psi))$, $x \in \mathbb{R}^n, \omega^k \in \Lambda(\mathbb{R}^n), \psi \in \triangle$.

These Lemmata give us a morphism of the associated bundles: $T \times S \longrightarrow S$.

Theorem 5 *Let X be a CW complex such that* $H^2(X, \mathbb{Z})$ *has no 2-torsion. Then Spinorbundles associated to given different Spin-structures of a principle* $SO(n)$*bundle, are isomorphic.*

Connections in Spinorbundles

Let (M^n, g) be an oriented, connected riemannian manifold, $(Q, \Pi, M^n; SO(n))$ a principle bundle of oriented orthonormal frames (Reperbundle).
There is a unique torsion free connection for M, as a covariant derivative of vector fields, the Levi- Civita- connection $\bigtriangledown$. Regarded as a connection in the bundle we have a $so(n)$ -1-form $Z : TQ \to so(n) =$ Liealgebra of $SO(n)$.

Definition 7 *A* $Spin^{\mathbb{C}}$***-structure*** *is a pair* (P, Λ) *such that*

- *P is a principle $Spin^{\mathbb{C}}$bundle over X and*
- *Λ is a map $P \to Q$ i.e. the following diagram commutes*

$$\begin{array}{ccc} PxSpin^{\mathbb{C}} & \longrightarrow & P \\ \downarrow{\scriptstyle \Lambda x\lambda} & & \downarrow{\scriptstyle \lambda} \\ QxSO(n) & \longrightarrow & Q \end{array}$$

where λ is the S^1 fibration of $Spin^{\mathbb{C}}$over $SO(n)$. And let $Spin^{\mathbb{C}} := Spin(n) \times S^1/\{+1, -1\} = Spin(n) \times_{\mathbb{Z}} S^1$.

Fix a connection A in the principle $U(1) = S^1 \cong SO(2)$ bundle $P_1 : A : TP_1 \to u(1) = i\mathbb{R}$ =Liealgebra of $U(1)$.
The two connections A, Z give a connection

$$Z \times A : T(Q\tilde{\times}P_1) \to so(n) \oplus i\mathbb{R}$$

this connection lifts to a connection

$$\widetilde{Z \times A} : T(P) \to Spin^{\mathbb{C}}(n)$$

since $\pi : P \to Q\tilde{\times}P_1$ and $p : Spin^{\mathbb{C}}(n) \to SO(n) \times S^1$ are 2-sheeted coverings.

$$\begin{array}{ccc} T(P) & \xrightarrow{\widetilde{Z\times A}} & Spin^{\mathbb{C}}(n) \\ \downarrow{\scriptstyle d\pi} & & \downarrow{\scriptstyle p_*} \\ T(Q)\tilde{\times}P_1 & \xrightarrow{Z\times A} & so(n) \oplus i\mathbb{R}^1 \end{array}$$

The **local form for the connection** can be described as follows.
A connection is characterized (local) by a section $s : U \in M \to Q$ by regarding $Z^s = Z \circ ds : TU \to Liealg(G)$. Let $e : U \in M^n \to Q$ be a local section in the frame bundle Q. Then $e = (e_1, ..., e_n)$ is a orthonormal n base of vectorfields defined on the open set U. With this the local connection is given by $Z^e = Z \circ de = \sum_{i<j} \omega_{ij} E_{ij} : TU \to so(n)$ where $\omega_{ij} = g(\bigtriangledown e_i, e_j)$ defining the Levi- Civita- connection and E_{ij} the standard base in $so(n)$.
Analog defining a section $s : U \to P_1$ follows that $e \times s$ is a local section in the principle $U(1)$ bundle $Q \times P_1$ that can be lifted in the 2-sheeted covering $\pi : P \to Q\tilde{\times}P_1$. With this we have for the local form of the connection in the Spinorbundle

$$\widetilde{Z \times A} = (\frac{1}{2} \sum_{i<j} \omega_{ij} e_i e_j, \frac{1}{2} A \circ ds)$$

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References

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