Spin Representations

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First I want to recall some definitions and notations that were introduced last time:
We denote by $\mathcal{C}_n = C(\mathbb{R}^n, -x_1^2 - \cdots - x_n^2)$ the Clifford algebra of the n-dimensional negativ definit real form and by $\mathcal{C}_n^c = C(\mathbb{C}^n, z_1^2 + \cdots + z_n^2)$ its complexification.
We have seen that $\mathcal{C}_n$ is a $\mathbb{Z}_2$-graded algebra: $\mathcal{C}_n = \mathcal{C}_n^0 \oplus \mathcal{C}_n^1$
The group $\mathrm{Pin}(n) \subset \mathcal{C}_n$ is generated (multiplicatively) by the elements of $S^{n-1} \subset \mathbb{R}^n$ and we define $\mathrm{Spin}(n) = \mathrm{Pin}(n) \cap \mathcal{C}_n^0$ *(i.e. it consists of all elements of* $\mathrm{Pin}(n)$ *with an even number of factors)*. There is a surjective group homomorphism $\lambda : \mathrm{Pin}(n) \to O(n)$ with $\lambda^{-1}(SO(n)) = \mathrm{Spin}(n)$ and $\ker(\lambda) = \{\pm 1\}$.
$\Delta_n := \mathbb{C}^{2^k}$ for $n = 2k, 2k+1$ is the space of complex n-spinors. It is a module over $\mathcal{C}_n^c$ because we have the spinor representation

$$\kappa_n : \mathcal{C}_n^c \xrightarrow{\sim} End(\Delta_n) \quad \text{for } n \text{ even, resp.}$$

$$\kappa_n : \mathcal{C}_n^c \xrightarrow{\sim} End(\Delta_n) \oplus End(\Delta_n) \xrightarrow{pr_1} End(\Delta_n) \quad \text{for } n \text{ odd}$$

Since $Spin(n) \subset \mathcal{C}_n \subset \mathcal{C}_n^c$ by restriction we get a representation of the group $Spin(n)$:

$$\kappa := \kappa_n|_{\mathrm{Spin}(n)} : \mathrm{Spin}(n) \to Aut(\Delta_n)$$

Proposition: The spinor representation is a faithful representation of the group $Spin(n)$.

Proof: For $n = 2k$ this is trivial, for $n = 2k+1$ we have $\Delta_{2k+1} = \Delta_{2k}$ (as vector spaces) and the diagram

$$\begin{array}{ccc} \mathrm{Spin}(2k) & \xrightarrow{\kappa_{2k}} & GL(\Delta_{2k}) \\ \downarrow & & \downarrow \\ \mathrm{Spin}(2k+1) & \xrightarrow{\kappa_{2k+1}} & GL(\Delta_{2k+1}) \end{array}$$

(where the vertical arrows denote the inclusion resp. the identity) commutes. Let $H := \ker(\kappa_{2k+1})$ and $h \in H \cap \mathrm{Spin}(2k)$, then it follows from the commutativity that $\kappa_{2k}(h) = 1$, hence $h = 1$ (since κ_{2k} is injective), i.e. the intersection is trivial.
λ is surjective and therefore the subgroup $\lambda(H)$ is normal in $SO(2k+1)$ *(general fact)*. Moreover one has *(by a similar argument as above)* $\lambda(H) \cap SO(2k) = \{E\}$, and we show

now $\lambda(H) = \{E\}$: For an element A of this group there exists a vector v_0 satisfying $A(v_0) = v_0$ *(in odd dimensions there always exists a real eigenvalue, and here it has to be one)*, and a $B \in SO(2k+1)$, such that $BAB^{-1} \in SO(2k)$. Since $\lambda(H)$ is normal, it follows that $BAB^{-1} \in \lambda(H) \cap SO(2k)$, hence $BAB^{-1} = E$ and finally $A = E$.
We have seen that λ is a twofold covering, and so the only remaining possibilities are $H = \{1\}$ or $H = \{1, -1\}$. But the element $-1 \in \mathrm{Spin}(2k+1)$ clearly isn't in the kernel of the spinor representation. qed

The Clifford multiplication

Since $\mathbb{R}^n \subset \mathcal{C}_n \subset \mathcal{C}_n^c$ we can regard a vector $x \in \mathbb{R}^n$ as an endomorphism of Δ_n and get the so called Clifford multiplication of vectors and spinors as a linear map

$$\mu : \mathbb{R}^n \otimes_{\mathbb{R}} \Delta_n \to \Delta_n,$$

defined by $\mu(x \otimes \psi) = \kappa_n(x)(\psi) =: x \cdot \psi$.

This multiplication can be extended to a homomorphism

$$\mu : \Lambda(\mathbb{R}^n) \otimes_{\mathbb{R}} \Delta_n \to \Delta_n.$$

For an element $w = \sum_{i_1 < \ldots < i_k} w_{i_1 \ldots i_k} e_{i_1} \wedge \ldots \wedge e_{i_k}$ we set:

$$w \cdot \psi = \sum_{i_1 < \ldots < i_k} w_{i_1 \ldots i_k} e_{i_1} \cdot \ldots \cdot e_{i_k} \cdot \psi.$$

A direct calculation yields the formula $(x \wedge w) \cdot \psi = x \cdot (w \cdot \psi) + (x \lrcorner w) \cdot \psi$, where $\lrcorner$ denotes the so called inner multiplication, defined by

$$x \lrcorner (e_{i_1} \wedge \ldots \wedge e_{i_k}) = \sum_{j=1}^{k} (-1)^{j-1} \langle x, e_j \rangle e_{i_1} \wedge \ldots \wedge \hat{e_{i_j}} \wedge \ldots \wedge e_{i_k}$$

$\mathrm{Spin}(n)$ acts on $\mathbb{R}^n$ via λ, and we extend this action to $\Lambda(\mathbb{R}^n)$ in the natural way. Then we have:

Proposition: The Clifford multiplication is equivariant w.r.t. the action of $Spin(n)$, i.e. for all $g \in \mathrm{Spin}(n), w \in \Lambda(\mathbb{R}^n)$ and $\psi \in \Delta_n$ we have

$$\kappa(g)(w \cdot \psi) = (\lambda(g)w) \cdot (\kappa(g)\psi).$$

Proof: We proceed by induction over the degree k of w. Let first be $k = 1$ und $w = x \in \mathbb{R}^n$, then one has:

$$\begin{aligned} \kappa(g)(x \cdot \psi) &= \kappa(g)\kappa_n(x)\psi = \kappa(g)\kappa_n(x)\kappa(g^{-1})\kappa(g)\psi = \kappa_n(gxg^{-1})\kappa(g)\psi \\ &= \kappa_n(\lambda(g)x)(\kappa(g)\psi) = (\lambda(g)x) \cdot (\kappa(g)\psi) \end{aligned}$$

(recall: $\lambda(g)x = gx\gamma(g)$ and $\gamma(g) = g^{-1}$ for $g \in \mathrm{Spin}(n)$)
Now we assume that the formula holds for all elements $w \in \Lambda(\mathbb{R}^n)$ of degree $\leq k$ and consider $w^{k+1} := x \wedge w^k$:

$$\begin{aligned}
\kappa(g)((x \wedge w^k) \cdot \psi) &= \kappa(g)(x \cdot (w^k \cdot \psi)) + \kappa(g)((x \lrcorner w^k) \cdot \psi) \\
&= (\lambda(g)x) \cdot (\kappa(g)(w^k \cdot \psi)) + \lambda(g)(x \lrcorner w^k) \cdot \kappa(g)\psi \\
&= (\lambda(g)x) \cdot ((\lambda(g)w^k) \cdot \kappa(g)\psi) + (\lambda(g)x \lrcorner \lambda(g)w^k) \cdot \kappa(g)\psi \\
&= ((\lambda(g)x) \wedge (\lambda(g)w^k)) \cdot \kappa(g)\psi = (\lambda(g)w^{k+1})\kappa(g)\psi
\end{aligned}$$

qed

Now consider the case $n = 2k$: Then the element $e_1 \dots e_{2k}$ lies in the centre of the algebra $\mathcal{C}_n^0$ *(since* $Z(\mathcal{C}_{2k}^0) = \mathbb{R} \oplus \mathbb{R}[e_1 \dots e_{2k}]$*, as we have seen last time)* and so commutes in particular with all elements of $\mathrm{Spin}(n) \subset \mathcal{C}_n^0$. Therefore the endomorphism

$$f = i^k \kappa(e_1 \dots e_{2k}) : \Delta_{2k} \to \Delta_{2k}$$

is an automorphism of the spinor representation, i.e. $f(\kappa(g)\psi) = \kappa(g)f(\psi)$. Moreover f is an involution, because of the relation $(e_1 \dots e_{2k})^2 = (-1)^k$, and so has eigenvalues ± 1. We decompose Δ_{2k} into the eigenspaces of f:

$$\Delta_{2k} = \Delta_{2k}^+ \oplus \Delta_{2k}^- \quad \text{where } \Delta_{2k}^\pm = \{\psi \in \Delta_{2k} : f(\psi) = \pm\psi\}$$

Definition: The elements of the spaces $\Delta_{2k}^\pm$ are called (positive resp. negative) Weyl spinors.

Proposition:

1. $\dim_{\mathbb{C}} \Delta_{2k}^+ = \dim_{\mathbb{C}} \Delta_{2k}^- = 2^{k-1}$
2. If $x \in \mathbb{R}^{2k}$ and $\psi^\pm \in \Delta_{2k}$, then the spinor $x \cdot \psi^\pm$ lies in $\Delta_{2k}^\mp$. Therefore the Clifford multiplication induces homomorphisms

$$\mu : \mathbb{R}^{2k} \otimes_{\mathbb{R}} \Delta_{2k}^\pm \to \Delta_{2k}^\mp.$$

Proof: From the relation $x(e_1 \dots e_{2k}) = -(e_1 \dots e_{2k})x$ in the algebra $\mathcal{C}_n$ it follows that $f(x \cdot \psi) = -x \cdot f(\psi)$, i.e. f and the Clifford multiplication with a vector x anti-commute. Therefore multiplication with $x \neq 0$ maps $\Delta_{2k}^\pm$ bijectively to $\Delta_{2k}^\mp$. This proves both assertions, since we know $\dim_{\mathbb{C}} \Delta_{2k} = 2^k$. qed

Irreducibility of the spinor representations

First we need some preparations:

Proposition: For all $n \in \mathbb{N}$ there is an algebra isomorphism $\mathcal{C}_n \cong \mathcal{C}^0_{n+1}$.

Proof: Choose an o.n.b. $e_1, \ldots, e_{n+1}$ of $\mathbb{R}^{n+1}$ and let $\mathbb{R}^n = \langle e_1, \ldots, e_n \rangle$. Define a map $f : \mathbb{R}^n \to \mathcal{C}^0_{n+1}$ by setting $f(e_i) = e_{n+1}e_i$ and extending linearly. One easily checks that f extends to an algebra homomorphism $\tilde{f} : \mathcal{C}_n \to \mathcal{C}^0_{n+1}$ and that this is an isomorphism. qed

Lemma: If V, W are two complex vector spaces with $\dim V > \dim W$, then every homomorphism of algebras $f : \mathrm{End}(V) \to \mathrm{End}(W)$ is trivial.

This follows immediately from the following

Theorem: The endomorphisms of a complex vector space form a simple algebra, i.e. there are no proper ideals.

Proof: Let V be a n dimensional complex vector space and $\mathcal{A} := \mathrm{End}(V)$. If $\mathcal{I} \neq \{0\}$ is an ideal in $\mathcal{A}$, there exist $\phi \in V$ and $a \in \mathcal{I}$, so that $a\phi = \psi \neq 0$. Since $\mathcal{I}$ is a two-sided ideal, for every two vectors $\phi_1, \phi_2 \in V$ with $\phi_1 \neq 0$ there is an element $b \in \mathcal{I}$ that maps ϕ_1 to ϕ_2.
Now choose a basis $\{e_1, \ldots, e_n\}$ of V and $a_i \in \mathcal{I}$ so that $a_i e_i = e_i$. Let $P_i \in \mathcal{A}$ be defined by $P_i(e_k) = \delta_{ik}e_k$, then $a_iP_i \in \mathcal{I}$ and $a_iP_i = P_i$, therefore $P_i \in \mathcal{I}$. But the sum of the P_i is 1 and $1 \in \mathcal{I}$ implies $\mathcal{I} = \mathcal{A}$. qed

Proposition: The representations $\Delta^{\pm}_{2k}$ of $\mathrm{Spin}(2k)$ are irreducible.

Proof: Lets assume that there exists a $\mathrm{Spin}(2k)$-invariant subspace $0 \neq W \subsetneqq \Delta^+_{2k}$. Consider the inclusions

$$\mathrm{Spin}(2k) \subset (\mathcal{C}^c_{2k})^0 \subset \mathcal{C}^c_{2k} = \mathrm{End}\left(\Delta^+_{2k} \oplus \Delta^-_{2k}\right).$$

The products $e_i \cdot e_j$ with $i < j$ are in $\mathrm{Spin}(2k)$ und so leave W invariant, on the other hand they generate the algebra $(\mathcal{C}^c_{2k})^0$ multiplicatively. Consequently we get a representation

$$f : (\mathcal{C}^c_{2k})^0 \to \mathrm{End}(W).$$

By the above proposition we have $(\mathcal{C}^c_{2k})^0 \cong \mathcal{C}^c_{2k-1} = \mathrm{End}(\Delta_{2k-1}) \oplus \mathrm{End}(\Delta_{2k-1})$ and since $\dim W < \dim \Delta^+_{2k} = 2^{k-1} = \dim \Delta_{2k-1}$ the representation f has to be trivial (according to the lemma above), but that's a contradiction. qed

With a similar argument one sees:

Proposition: The representation Δ_{2k+1} of $\mathrm{Spin}(2k+1)$ is irreducible.

Proof: Here one has the inclusions

$$\mathrm{Spin}(2k+1) \subset \left(\mathcal{C}^c_{2k+1}\right)^0 \subset \mathcal{C}^c_{2k} = \mathrm{End}\left(\Delta_{2k+1}\right) \oplus \mathrm{End}\left(\Delta_{2k}\right).$$

and we assume that $0 \neq W \subsetneqq \Delta_{2k+1}$ is a Spin$(2k)$-invariant subspace. Like before we get a representation

$$f : \left(\mathcal{C}^c_{2k+1}\right)^0 \to \mathrm{End}(W),$$

which has to be trivial, because $\left(\mathcal{C}^c_{2k+1}\right)^0 \cong \mathcal{C}^c_{2k} = \mathrm{End}(\Delta_{2k})$ and $\dim W < \dim \Delta_{2k+1} = \dim \Delta_{2k}$. Again that's a contradiction. qed

Unitarity

Proposition: In Δ_n there exists a positive definite hermitean inner product with the additional property

$$(x \cdot \psi, \varphi) + (\psi, x \cdot \varphi) \quad \text{for } x \in \mathbb{R}^n,\ \varphi, \psi \in \Delta_n.$$

The representation $\kappa : \mathrm{Spin}(n) \to GL(\Delta_n)$ is a unitary representation with respect to this inner product.

Proof: The group Pin(n) is a compact topological group, and so any finite dimensional representation is unitary w.r.t. a suitable inner product. If the representation is irreducible, this product is determined uniquely up to a scalar factor. Let $\langle .,. \rangle$ be such a product for the spinor representation of Pin(n). For $x \in S^{n-1}$ we have $\kappa(x)^* = \kappa(x)^{-1} = -\kappa(x)$, and by linearity the relation $\kappa(x)^* = -\kappa(x)$ holds also for all $x \in \mathbb{R}^n$. The claimed formula follows. Because of the uniqueness it doesn't matter that we startet with Pin(n) instead of Spin(n). qed

Proposition: For $n \geq 3$ the representation $\kappa : \mathrm{Spin}(n) \to U(\Delta_n)$ is a representation in the special unitary group $SU(\Delta_n)$ of the space of n-spinors, i.e. $\det(\kappa(g)) = 1$ for all $g \in \mathrm{Spin}(n)$.

Proof: That is not a special property of the representation, but follows from properties of the group Spin(n) itself. Consider the group homomorphism

$$f : \mathrm{Spin}(n) \to S^1,\ f(g) = \det(\kappa(g)).$$

Since Spin(n) is simply connected, there exists a lift $F : \mathrm{Spin}(n) \to \mathbb{R}$ (universal covering of S^1), which is also a group homomorphism, and $f(g) = e^{2\pi i F(g)}$. Since Spin(n) is compact, the subgroup $F(\mathrm{Spin}(n)) \subset \mathbb{R}$ is contained in a bounded interval and so has to be trivial. Hence $F \equiv 0$ and $f \equiv 1$. qed