The Theorem of Gel'fand-Graev

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Abstract

In this talk I want to show that there is a decomposition of the space squareintegrable cusp forms into irreducible invariant subspaces.

1 Notations

Let $G = \mathrm{SL}_2(\mathbb{R})$ the group of $\mathbb{R}$-valued points of the algebraic group $\mathrm{SL}_2/\mathbb{Q}$ and $\Gamma \subset G$ a discrete subgroup of G (for example a FUCHSian group of the 1st kind, or an arithmetic subgroup). As usual we want denote by $K = \mathrm{SO}(2)$ and the upper halfplane

$$\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\} = G/K \tag{1}$$

In the following we need the notion of function spaces $L^p(G)$, and of smooth functions $C^\infty(G)$ (resp. the compactly supported smooth functions $C_c^\infty(G)$) on the LIE group G in the usual sense via HAAR measure. For abitrary discrete subgroups $\Gamma \subset G$ the space $\Gamma \setminus G$ is not a manifold, hence we have to define $L^2(\Gamma \setminus G)$ by

$$L^2(\Gamma \setminus G) := \left\{ f \in L^2(G) : f(\gamma g) = f(g) \text{ for } \gamma \in \Gamma, g \in G \right\} \tag{2}$$

and analogue for the smooth functions

$$C^\infty(\Gamma \setminus G) := \{ f \in C^\infty(G) : f(\gamma g) = f(g) \text{ for } \gamma \in \Gamma, g \in G \} \tag{3}$$

and by $L^2_{\mathrm{cusp}}(\Gamma \setminus G) \subset L^2(\Gamma \setminus G)$ the space of squareintegrable functions which are allmost everywhere cuspidal. Sometimes the notion $L^2_{\mathrm{cusp}}(\Gamma \setminus G) = {}^0L^2(\Gamma \setminus G)$ is also used.
Further we denote by

$$L^2_\infty(\Gamma \setminus G) := L^2(\Gamma \setminus G) \cap C^\infty(\Gamma \setminus G) \tag{4}$$

the space of C^∞-functions which are squareintegrable.

2 Basic Results

Lemma 2.1. *The subspace* $L^2_{\mathrm{cusp}}(\Gamma\backslash G) \subset L^2(\Gamma\backslash G)$ *is closed in* $L^2(\Gamma\backslash G)$ *for* $p \geq 1$.

Proof. See [Bor97, prop. 8.2 on p.75]. $\square$

Definition 2.2 (Siegel set). Let (P, A) a (normal) P pair and N the unipotent radical of $P.A$, a SIEGEL set (w.r.t. (P, A)) in G is a subset of the form

$$\mathfrak{S} = \omega.A_t K \qquad \text{for } \omega \subset N \text{ compact } A_t \, \{a \in A | a^\alpha > t\} \tag{5}$$

Lemma 2.3 (Gel'fand, Graev, Piatetski-Shapiro 1968, Godement 1966/70). *Fix a* $p \geq 1$ *for* $p \in \mathbb{R}$ *and* $\phi \in C^1_c(G)$, *then there exists a constant* $C_\phi > 0$ *such that*

$$|(f * \phi)(x)| \leq C_\phi \|f\|_p \qquad x \in \Gamma \setminus G \tag{6}$$

for all $f \in L^2_{\mathrm{cusp}}(\Gamma \setminus G)$.

Lemma 2.4. *Let* $p \geq 1$ *and* $\phi \in C^\infty_c(G)$, *and let* P *be a (cuspidal) parabolic subgroup for* Γ. *Then there is a constat* $c_\phi > 0$ *such that*

$$|(f * \phi)(x)| \leq c_\phi \|f\|_p a(x)^{2\rho/p} \qquad x \in NA_t K \tag{7}$$

for all $f \in L^p(\Gamma \setminus G)$. *Where* $G = NAK$ *is the Iwasawa decomposition and the character* $\rho \in X(A)$ *ist defined by the rule* $a^\rho = \|a.e_P\|$ *where* e_p *is a unit vector which is stable under* P.

Proof. (Lemma 2.4)
The function f is Γ_N-invariant on the left, so we may assume x to be contained in a SIEGEL set

$$\mathfrak{S}_{P,t} = \omega A_t K \tag{8}$$

where ω is relatively compact subset of N containing a fundamental domain for $\Gamma_N = N \cap \Gamma$. Let C be a compact set containing $\mathrm{supp}\,(\phi)$. Then we have

$$KC \subseteq C_N C_A K \tag{9}$$

with $C_A \subset A$ $C_N \subset N$ compact, hence

$$\begin{aligned} xC &\subseteq \omega a(x) KC \\ &\subseteq \omega a(x) C_N C_A K \\ &\subseteq a(x) \left(a(x)^{-1} \omega a(x) C_N\right) a(x)^{-1} a(x) C_A K \end{aligned} \tag{10}$$

By definition we have

$$(f * \phi)(x) = \int_G f(y)\phi(y^{-1}x)\,dy \tag{11}$$

Whence

$$|f * \phi(x)| \leq \|\phi\|_\infty \int_{xC} |f(y)| \tag{12}$$

Since $a(x)^{-2\rho} \prec 1$ on $\mathfrak{S}_{P,t}$, the set $a(x)^{-1}\omega a(x)C_N \subset N$ is bounded and hence contained in a compact set D. Therefore

$$xC \subset a(x)Da(x)^{-1}a(x)C_AK \tag{13}$$

The automorphism $n \mapsto a(x)na(x)^{-1}$ of $N \cong \mathbb{R}$ is the dilation $a(x)^{2\rho}$. Therefore

$$\mu_{\mathbb{R}}\left(a(x)Da(x)^{-1}\right)a(x)^{2\rho}\mu_{\mathbb{R}}(D) \tag{14}$$

Let $Q(x) = D.a(x).C_A.K$. Because $|f(y)|$ is left-invariant under $\Gamma_N = \Gamma \cap N$, we can write

$$\int_{xC} |f(y)|\,dy \leq a(x)^{2\rho} \int_{Q(x)} dy \tag{15}$$

Now we aplly HÖLDER's inequality with $\frac{1}{p} + \frac{1}{q} = 1$

$$\int_{Q(x)} |f(y)\,dy \leq \underbrace{\left(\int_{Q(x)} |f(y)|^p\right)^{1/p}}_{\leq \|f\|_p} \left(\int_{Q(x)} \quad dy\right)^{1/q} \tag{16}$$

The 2nd integral is $\asymp a(x)^{-2\rho/q}$. Altogether, we have

$$\int_{Q(x)} |f(y)|\,dy \leq c\|f\|_p a(x)^{2\rho-2\rho/q} = c\|f\|_p a(x)^{2\rho/p} \tag{17}$$

with a constand c independent of f, and the lemma follows. □

Proof. (Lemma 2.3)
See in [Bor97, theor. 9.2 on p. 81] or in [Bum98, prop. 3.2.3 on p. 285]

Let P be a cuspidal parabolic subgroup for Γ and $\mathfrak{S}$ the SIEGEL[1] set relative to P. By Lemma 2.4 there exists a constant $c(\phi)$ such that

$$|(f * \phi)(x)| \leq c(\phi)\,\|f\|_p\,a(x)^{2\rho/p} \qquad x \in \mathfrak{S}, f \in L^p(\Gamma \setminus G) \tag{18}$$

[1] Was ist eine Siegel menge

For $D \in \mathfrak{U}(\mathfrak{g})$ we have $D(f * \phi) = f * D\phi$. Hence we have also that

$$|D(f * \phi)(x)| \leq c(\phi, D) \|f\|_p \, a(x)^{2\rho/p} \qquad x \in \mathfrak{S}, f \in L^p(\Gamma \setminus G) \tag{19}$$

The contant term for a Parabolic subsgroup $P = N.A$ of continuous Γ_N-invariant $f \in L^1_{\text{loc}}(G)$ is defined by

$$f_P(g) = \int_{\Gamma \setminus \Gamma_N} f(ng) \, dn \qquad g \in G \tag{20}$$

By this definition an since $a(nx) = a(x)$ for $n \in N$ and $x \in G$, we have further

$$|D(f * \phi)_P(x)| \leq c(\phi, D) \|f\|_p \, a(x)^{2\rho/p} \qquad x \in \mathfrak{S}, f \in L^p(\Gamma \setminus G) \tag{21}$$

If we assume now that f is cuspidal at P, that is $f_P = 0$. Then $(f * \phi)_P = 0$. We can show by applying the previous results to a basis of the LIE-algebra that there exists a constant $c'(\phi)$ (not depending on f.

$$|(f * \phi)(x)| \leq c'(\phi) \|f\|_p \, a(x)^{2\rho/p} \qquad x \in \mathfrak{S}, f \in L^p(\Gamma \setminus G) \tag{22}$$

Now we can iterate this argument, and it follows that for a given $m \in \mathbb{N}$ there exists a constant $c(m, \phi)$

$$|(f * \phi)(x)| \leq c(m, \phi) \|f\|_p \, a(x)^{2\rho/p} . a(x)^{m2\rho} \qquad x \in \mathfrak{S}, f \in L^p(\Gamma \setminus G) \tag{23}$$

Therefore, if $m \geq 2$

$$a(x)^{2\rho/p - m2\rho} \leq 1 \qquad x \in \mathfrak{S} \tag{24}$$

Formula 18 follows from the both previous formulae. If now $f \in L^2_{\text{cusp}}(\Gamma \setminus G)$, then this estimation is valid on a SIEGEL set relative to any parabolic P for Γ. Since $\Gamma \setminus G$ is covered by finitly many such sets Lemma 2.3 is proved. $\square$

Theorem 2.5. Let $p \in \mathbb{R}$, $p \geq 1$, and $\phi \in C^1_c(G)$. Then the operator
$. * \phi$ on $L^2_{\text{cusp}}(\Gamma \setminus G)$ is a compact operator.

Proof. Let $D \in \mathfrak{U}(\mathfrak{g})$. Since $D(f * \phi) = f * D\phi$, Lemma 2.3 implies that there is a constant $c_{\phi,D} > 0$ such that

$$|D(f * \phi)(x)| \leq c_{\phi,D} \|f\|_p \tag{25}$$

for $f \in L^2_{\text{cusp}}(\Gamma \setminus G)$. Now we fix an open subset $U \subset \Gamma \setminus G$ which is homeomorphic to a (small) coordinate neighborhood in G, with the coordinates x_1, x_2, x_3. The partial derivative with respect to each x_i of any order are the linear combination of elements in $\mathcal{U}(\mathfrak{g})$ with smooth coefficients. Hence the Formula 25 is valid for arbitrary partial derivatives in $x \in U$, too.
Let now $\{f_n\} \subset L^2_{\text{cusp}}(\Gamma \setminus G)$ for $n \in \mathbb{N}$ a sequence, with $\|f_n\| \le a$ for some constant $a > 0$. Then the sequence of the convolutions $f_n * \phi$ is uniformly bounded in U, too. Hence the sequence $f_n * \phi$ and their derivatives form an equicontiuous family. By the Theorem of ARZELA-ASCOLI this family has a uniformly convergent subsequence.
Further (via diagonal processes) we can find a countable covering of $\Gamma \setminus G$ by sets, which has the same properties as the U from above. Therefore we can find a subsequence $f_{n'}$ of f_n such that $f_{n'} * \phi$ converges locally uniformly to a function f, which is continuous bounded by a. Since $\Gamma \setminus G$ has finite measure, it follows that $f \in L^p(\Gamma \setminus G)$ and the subsequence $f_{n'} \to f$ in L^p. By Lemma 2.1 we know that $L^2_{\text{cusp}}(\Gamma \setminus G)$ is closed in $L^2(\Gamma \setminus G)$, hence $f \in L^2_{\text{cusp}}(\Gamma \setminus G)$. ☐

Remark 2.6. Let $\phi \in C^1_c(G)$. Then the operator $\,.\, * \phi$ is a HILBERT-SCHMIDT operator of trace class on $L^2_{\text{cusp}}(\Gamma \setminus G)$.

We have $L^2_{\text{cusp}}(\Gamma \setminus G)$ and $L^2(\Gamma \setminus G)$ defined as above then we want to define a scalar product on $L^2(\Gamma \setminus G)$ by

$$(\phi, \psi) := \int_{\Gamma \setminus G} \phi(g).\overline{\psi(g)}\, dg \qquad \text{for } \phi, \psi \in L^2(\Gamma \setminus G) \tag{26}$$

Definition 2.7 (DIRAC sequence). A sequence of compactly supported functions $(\alpha_n)_{n \in \mathbb{N}}$ $\alpha \in C^\infty_c(G)$ is called DIRAC sequence if $\alpha_n(x) \ge 0$ for all $x \in G$ and

$$\operatorname{supp} \alpha_n \to \{1\} \qquad \text{for } n \to \infty \tag{27}$$

$$\int_G \alpha_n(x)\, dx = 1 \qquad \text{for all } n \in \mathbb{N} \tag{28}$$

Then for any $f \in C^\infty_c(G)$, the sequence $f * \alpha_n$ converges absolute and uniform on compact set to f and for each derivative Df for $D \in \mathcal{U}(\mathfrak{g})$.

Lemma 2.8. *Let $\alpha, \alpha_n \in C^\infty_c(G)$ for $n \in \mathbb{N}$, and assume that the $(\alpha_n)_{n \in \mathbb{N}}$ form a DIRAC sequence, and let $f \in L^2(\Gamma \setminus G)$. Then*

1. *The convolution* $. * \alpha$ *by* α *is a bounded operator in the* HILBERT *space* $L^2(\Gamma \setminus G)$ *with* $. * \alpha^*$ *as adjoint operator.*

2. *The function*

$$\begin{aligned} r_. : G &\rightarrow L^2(\Gamma \setminus G) \\ x &\mapsto r_x f \end{aligned} \tag{29}$$

   is continuous.

3. *The sequence* $f * \alpha_n \rightarrow f$ *converges in* $L^2(\Gamma \setminus G)$ *for* $n \rightarrow \infty$*. In particular* $L^2_\infty(\Gamma \setminus G) \subset L^2(\Gamma \setminus G)$ *dense.*

Proof. See [Bor97, 13.1 Lemma, page 145]. □

Let (π, V) a continuous representation of G into V then we can - under some assumptions on the meassure on G - we can extend π to $C_c^\infty(G)$ by

$$\pi(\alpha)v = \int_G \alpha(x)\pi(x)v\,dx \qquad \text{for } \alpha \in C_c^\infty(G) \tag{30}$$

A representation is called *unitary* if it leaves the scalar product on V invariant.
We denote by $\check{\alpha}$ the mapping

$$\check{\alpha} : g \mapsto \alpha(g^{-1}) \tag{31}$$

$$\alpha^* : g \mapsto \overline{\alpha(g^{-1})} \tag{32}$$

Lemma 2.9 (C^∞ representations). *If* $(\pi, V) = (r, C^\infty(G))$ *the right regular representation, then* $\pi(\alpha)$ *is the convolution* $. * \check{\alpha}$ *by* $\check{\alpha}$*. This extension satisfies the rule*

$$\pi(\alpha * \beta) = \pi(\alpha) \circ \pi(\beta) \qquad \textit{for } \alpha, \beta \in C_c^\infty(G) \tag{33}$$

Let $\{\alpha_n\}_{n\in\mathbb{N}}$ *be a* DIRAC *sequence then*

$$\pi(\alpha_n)v \rightarrow v \tag{34}$$

If (π, V) *is unitary then the adjoint of* $\pi(\alpha)$ *is* $\pi(\alpha^*)$*.*

3 The Discrete Decomposition of $\mathrm{L}^2(\Gamma \setminus \mathrm{G})$

Lemma 3.1 (Gel'fand, Graev, Piatetski-Shapiro 1968, Godement 1966/70). *Let (π, E) be a unitary representation of G and $\{\alpha_n\}_{n\in\mathbb{N}}$ be a* DIRAC *sequence with compact operators $\pi(\alpha_n)$. Then E is a* HILBERT *direct sum of irredudicible representations, with finite multiplicities.*

Proof.

a) Existence of minimal G-invariant Subspaces First we want to show that every closed G-invariant subspace $0 \neq W \subseteq E$ contains a minimal closed G-invariant subspace, i.e. the subspace which is minimal with respect to the partial ordering given by inclusion. By Lemma 2.8 and Lemma 2.9, there is a $i \in \mathbb{N}$ such that $\pi(\alpha_i) \mid_W \neq 0$. Now let $\lambda \neq 0$ a eigenvalue of $\pi(\alpha_i)$ in W, and let $0 \neq M \subseteq W$ the corresponding (finite dimensional) eigenspace in W.

Let

$$N = \bigcap_{\substack{V \subseteq W \\ GV \subseteq V \\ M \cap V \neq \emptyset}} M \cap V \tag{35}$$

b) Decomposition into Subspaces Now we want to show that E has a discrete decomposition in closed irreducible G-invariant subspaces. Let

$$S = \{F \in E : GF \subseteq F, F \neq 0 \text{ closed, has a discr. decomp.}\}$$

By part a) $S \neq \emptyset$. The set S is a partialy ordered by the relation $X \leq Y$ if $X \subseteq Y$ and the discrete decomposition of X extends to a discrete decomposition of Y. This ordering is a inductive ordering.

Assumption: Let $W \subset E$ be the space underlying a maximal element. Because $W \neq E$ there exists by part a) a minimal closed nonzero G-invariant subspace in $W^\perp \subset E$. Hence W is not maximal and we have a contradiction.

c) Finiteness of the Multiplicities The last part which remains to show is the fact that the multiplicities are finite. Let (σ, F) be an irreducible unitary representation of multiplicity n_σ occuring in the unitary representation E.

Choose now a $i \in \mathbb{N}$ such that $\sigma(\alpha_i) \mid_F \neq 0$ and let $\lambda \neq 0$ be an eigenvalue of $\sigma(\alpha_i)$ whith multiplicity m. Then we see because all eigenspaces have finite dimension that on the one hand the dimension of the eigenspace of $\pi(\alpha_i)$ in E with eigenvalue λ is finite. On the other hand the dimension is at least $m\, n_\sigma$. Hence the multiplicity n_σ is finite.

□

There are some prooves of this theorem which are using using different arguments.

Theorem 3.2 (I.M. Gel'fand, M.I. Graev 1968). The space $L^2_{\mathrm{cusp}}(\Gamma \setminus G)$ decomposes into a Hilbert direct sum of closed irreducible G-invariant subspaces with finite multiplicities.

Proof. In the right regular representation $(r, L^2_{\mathrm{cusp}}(\Gamma \setminus G))$, the endomorphism $\pi(\alpha) = *\alpha^\vee$ for $\alpha \in C_0^\infty(G)$. This operator is compact by theorem 2.5. Hence we can apply Lemma 3.1 which shows the theorem. □

Theorem 3.3. The space of cusp forms decomposes into a HILBERT direct sum

$$L^2_{\mathrm{cusp}}(\Gamma \setminus G) = \overline{\bigoplus_{\pi \in \widehat{G}} H_\pi^{m_c(\pi)}} \qquad (36)$$

where H_π is the representation space on which the group acts by the charackter π and $m_c(\pi)$ is the multiplicity of it in $L^2_{\mathrm{cusp}}(\Gamma \setminus G)$.

References

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