Automorphic forms on $\mathrm{GL}_2(\mathbb{A})$ and their connection to the classical theory

A talk given at Prof. W. Müller's seminar on spectral theory and automorphic forms

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Recall that $\mathbb{H} \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$. We have introduced classical automorphic forms as functions

$$f : \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2) \to \mathbb{C}$$

which have a certain transformation behaviour under the action from the left of a discrete subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$. If this behaviour were invariance, then we could consider these functions as functions on

$$\Gamma \setminus \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$$

Since the transformation behaviour is not invariance, automorphic forms are not quite the same as functions on this quotient. They can be viewed as "multi-valued functions" or, more precisely, as global sections of line bundles on this quotient. This we saw in Jörn's talk. Another thing we saw in his talk was an important redefinition of the notion of automorphic form. Instead it being a function on $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$ transforming under Γ from the left, an automorphic form was defined to be a function on $\Gamma \setminus \mathrm{SL}_2(\mathbb{R})$, transforming under $\mathrm{SO}(2)$ from the right. This new notion permits powerful generalizations, which include for example Maass forms.

Today we will take the latter definition of an automorphic form and will yet again generalize it to a new notion, which is the one used in the current theory. For this purpose we need to introduce the ring of adeles of $\mathbb{Q}$.

1 The p-adic numbers and the ring of adeles

In this section we will give a quick definition of $\mathbb{Q}_p$ and $\mathbb{A}$, stating only some of the most important facts and omitting most proofs. These can be read in almost any book on algebraic number theory, as f.e. [Ne] or [Mi].

Recall that an crucial property of the group Γ is that it is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$. This comes from the fact that $\mathbb{Z}$ is a discrete subset of $\mathbb{R}$. We want to find an object, which we call $\mathbb{A}$, with the property that $\mathbb{Q}$ is a discrete subset of $\mathbb{A}$. This object will be the ring of adeles of $\mathbb{Q}$.

1.1 The p-adic numbers

The completions of $\mathbb{Q}$.
It is well known that $\mathbb{R}$ is the complition of $\mathbb{Q}$ with respect to the topology of $\mathbb{Q}$ defined by the absolute value $x \mapsto |x|$. The absolute value satisfies three important properties

1. $|x| > 0$ for $x \neq 0$
2. $|xy| = |x||y|$
3. $|x + y| \leq |x| + |y|$

A map $\mathbb{Q} \to \mathbb{R}$ with these properties is called an **absolute value**. The standard absolute value is not the only one on $\mathbb{Q}$. For any prime p we can write a rational number $x \in \mathbb{Q}$ uniquely as $p^{v_p(x)}\frac{a}{b}$, where a and b are coprime and not divisible by p, and we can define the map

$$\mathrm{ord}_p : \mathbb{Q} \to \mathbb{Z}, \quad x = p^{v_p(x)}\frac{a}{b} \mapsto v_p(x)$$

which is called the "order of p in x". Then one easily checks that

$$|\ |_p : \mathbb{Q} \to \mathbb{R}, \quad x \mapsto p^{-\mathrm{ord}_p(x)}$$

is an absolute value on $\mathbb{Q}$.

Now we have an absolute value $|\ |_p$ for any prime p. We denote the standard absolute value by $|\ |_\infty$ (for reasons which we do not want to explain). It is of course reasonable to consider two absolute values equivalent, if they induce the same topology on $\mathbb{Q}$. The following theorem tells us that we have just enlisted all absolute values of $\mathbb{Q}$ up to equivalence

Theorem (Ostrowski): If $|\ |$ is a non-trivial absolute value on $\mathbb{Q}$ then $|\ |$ is equivalent to either $|\ |_\infty$ or to $|\ |_p$ for a prime p.

We denote the completion of $\mathbb{Q}$ with respect to $|\ |_p$ by $\mathbb{Q}_p$ and sometimes we would refer to $\mathbb{R}$ as $\mathbb{Q}_\infty$.

Properties of $|\ |_p$.
The absolute values $|\ |_p$ have two important features which distinguish them from $|\ |_\infty$. First, instead of the triangle inequality $|x+y| \leq |x|+|y|$ they satisfy the stronger inequality $|x+y| \leq \max(|x|,|y|)$. This property, immediately obvious from the definition of ord_p, gives $|\ |_p$ the name **nonarchimedian valuation**, because one immediately sees that for $m \in \mathbb{Z}$

$$|m|_p = |1+1+\cdots+1|_p \leq |1|_p = 1$$

which has the effect that the Archimedian axiom

$$\forall a, b \in \mathbb{Q}\ \exists n \in \mathbb{Z}\ |a| \leq |nb|$$

is no longer valid.

Second, the image of $|\ |_p$ is a discrete subset of $\mathbb{R}$ (it is the discrete set $\mathrm{im}(\mathrm{ord}_p) = \mathbb{Z}$ mapped under the continous map $x \mapsto p^{-x}$). This propery gives $|\ |_p$ the name **discrete valuation**. It has a crucial effect on the topology of $\mathbb{Q}_p$ – it is totally disconnected, which means that every point has a neighbourhood basis of sets which are both open and closed. Consider for example the point $0 \in \mathbb{Q}_p$. A neighbourhood basis is given by the sets

$$U_n = \underbrace{\{x \in \mathbb{Q}_p |\ |x|_p \leq p^{-n}\}}_{\text{closed}} = \underbrace{\{x \in \mathbb{Q}_p |\ |x|_p < p^{-n+\frac{1}{2}}\}}_{\text{open}}$$

and the equality of the two terms comes from the fact that $|\ |_p$ takes no value between p^{-n} and p^{-n+1}. We see that the topology of $\mathbb{Q}_p$ is completely different from the topology on $\mathbb{R}$, which has all the connectedness properties one could wish for. Yet, it has two very important features: It is Hausdorff, which makes limits of Caychy sequences unique and it is locally compact, which gives us a unique Haar measure.

An algebraic construction of $\mathbb{Q}_p$.
So far we have defined the fields $\mathbb{Q}_p$ as completions of $\mathbb{Q}$ with respect to the topology induced by the valuations $|\ |_p$. This of course gives us the well known construction of a completion of a topological space – it is the set of all Cauchy series modulo those converging to zero. We want to reformulate this construction in algebraic terms. This will give us a better understanding of the fields $\mathbb{Q}_p$ and will allow us to perform computations very easily.

We define the set

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p |\ |x|_p \leq 1\} = \{x \in \mathbb{Q}_p |\ \mathrm{ord}_p(x) \geq 0\}$$

observe that this set is a ring, thanks to the non-archimedian property of $|\ |_p$. This ring is called the ring of p-adic integers. Its units are the set

$$U_p = \{x \in \mathbb{Q}_p |\ |x|_p = 1\} = \{x \in \mathbb{Q}_p |\ \mathrm{ord}_p(x) = 0\}$$

It is obvious that the set

$$\mathfrak{m} = \{x \in \mathbb{Q}_p |\ |x|_p < 1\} = \{x \in \mathbb{Q}_p |\ \mathrm{ord}_p(x) > 0\}$$

is an ideal in $\mathbb{Z}_p$ and since it contains all elements except the units, it is a unique maximal ideal (i.e. $\mathbb{Z}_p$ is a local ring). Because our valuation $|\ |_p$ is discrete, every element $x \in \mathfrak{m}$ has $|x|_p \leq p^{-1}$, which implies $x = p\tilde{x}$ with $|\tilde{x}|_p \leq 1$ and we see that the ideal $\mathfrak{m}$ is a principal ideal generated by the element $p \in \mathbb{Z}_p$. The ring $\mathbb{Z}_p$ is an example of a discrete valuation ring.

Now we have the following

Proposition: Choose a set S of representatives for $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$ (f.e. $0, 1, 2, ...p-1$). Then the series

$$a_{-n}p^{-n} + \cdots + a_0 + a_1 p + \cdots a_m p^m + \cdots, \quad a_i \in S$$

is a Cauchy series, and every Cauchy series is equivalent to exactly one of this form.

Proof: Let $s_N = \sum_{i=-n}^{i=N} a_i p^i$ be the partial sums of the series. First for $M > N$

$$|s_M - s_N|_p \leq p^{N+1}$$

which shows that this is indeed a Cauchy series. Next let $x \in \mathbb{Q}_p$. Then there exists an N with $x = p^N y$ with $y \in \mathbb{Z}_p$. Now by the choice of S there is an $a_0 \in S$ s.t. $a_0 + p\mathbb{Z}_p = y + p\mathbb{Z}_p$. So

$$y = a_0 + py_1$$

with $y_1 \in \mathbb{Z}_p$. Playing the same game with y_1 gives us $y_1 = a_1 + py_2$ or

$$y = a_0 + pa_1 + p^2 y_2$$

Iterating this process gives us for y the series expansion

$$y = \sum_{i=0}^{\infty} a_i p^i$$

and thus

$$x = \sum_{i=-N}^{\infty} a_{i+N} p^i$$

For the uniqueness suppose there were two series of this type converging to x. Then their difference is again of this type and converges to zero. But the p-adic norm of such a series is equal to p^N where N is the first (possibly negative) index with $a_N \neq 0$. This means that the only sequence with norm zero is the trivial sequence.

$\square$

This proposition tells us, that the elements of $\mathbb{Q}_p$ are in $1-1$ correspondence with such series. Expressing elements of $\mathbb{Q}_p$ in terms of this series gives a very easy way of computing. Addition and multiplication of elements translates into addition and multiplication of series. The order of an element is N, where N is the smallest integer with $a_N \neq 0$. This means that the elements of $\mathbb{Z}_p$ are exactly the limits of those series which have no negative powers of p, and the elements of $\mathfrak{m} = p\mathbb{Z}_p$ are those elements in $\mathbb{Z}_p$ for which $a_0 = 0$.

Another very important property, which is obvious from the series, is that every $x \in \mathbb{Q}_p$ has the form

$$x = p^{-\mathrm{ord}(x)}\tilde{x}, \quad \tilde{x} \in U_p$$

For this reason one calls the set $\{p^n | n \in \mathbb{Z}\}$ the "spine" of $\mathbb{Q}_p$. It is now obvious that $\mathbb{Q}_p$ is the quotient field of $\mathbb{Z}_p$ (observe that $\mathbb{Z}_p$ is a domain, because it admits a valuation) and we can write it as

$$\mathbb{Q}_p = \bigcup_{n=0}^{\infty} p^{-n}\mathbb{Z}_p = \bigcup_{n=-\infty}^{n=+\infty} p^n U_p$$

Now we want to use the $1-1$ correspondence between elements of $\mathbb{Z}_p$ and Cauchy series $\sum_{i=0}^{\infty} a_i p^i$ to give an algebraic construction of the ring $\mathbb{Z}_p$. Let $x \in \mathbb{Z}_p$ and consider the unique Cauchy series $\sum a_i p^i$ whose limit x is. If we write the partial sums $s_n = \sum_{i=0}^{n} a_i p^i$ then $x = \lim_{n\to\infty} s_n$. Note that all s_n are integers, but their limit x of course need not be. The element x is uniquely identified by the sequence $(s_i)_{i\in\mathbb{N}}$ whose limit it is. Any of the partial sums s_n can be mapped to $\mathbb{Z}/p^m Z$ for an arbitrary m. If $m > n$, then the image of s_n in $\mathbb{Z}/p^m Z$ is different from the image of the n-th partial sum of any other Cauchy sequence of this type. Thus it uniquely identifies s_n. This is in particular true for the image of s_n in $\mathbb{Z}/p^{n+1}Z$. But this image carries even more information. For $n' < n$ there is a natural projection $\mathbb{Z}/p^{n'}\mathbb{Z} \leftarrow \mathbb{Z}/p^n\mathbb{Z}$ and if we map s_n to $\mathbb{Z}/p^{n+1}Z$ and then project it down to $\mathbb{Z}/p^{n'+1}\mathbb{Z}$ we get the same element as if we map $s_{n'}$ directly to $\mathbb{Z}/p^{n'+1}\mathbb{Z}$. In other words, the image of s_n in $\mathbb{Z}/p^{m+1}\mathbb{Z}$ coincides with the image of s_m in $\mathbb{Z}/p^{m+1}\mathbb{Z}$ for all $m < n$. One could visualize the process of projecting down one step as removing the highest power of p from the series $\sum_{i=0}^{n} a_i p^i$.

We just saw that the beginning $s_0, ..., s_n$ of the sequence $\{s_i\}_{i\in\mathbb{N}}$ corresponds uniquely to a sequence $\bar{s}_0, ..., \bar{s}_{n-1}$ with the properties

1. $\bar{s}_i \in \mathbb{Z}/p^i\mathbb{Z}$

2. the image of $\bar{s}_i$ in $\mathbb{Z}/p^j\mathbb{Z}$ is s_j for $j < i$

It is also very easy to see that the converse is also true, namely any such sequence comes from the first n partial sums $s_1, ...s_n$ of a Cauchy series $\sum a_i p^i$. The argument is the same as in the proof of the above proposition. Now it is clear that an infinite sequence $\{\bar{s}_i\}$ with the above properties corresponds uniquely to the sequence $\{s_i\}$ of partial sums of a Cauchy series. Thus the ring $\mathbb{Z}_p$ can be identified with the ring of sequences $\{\bar{s}_i\}$. This ring is called the projective limit of the system

$$\mathbb{Z}/p\mathbb{Z} \leftarrow \mathbb{Z}/p^2\mathbb{Z} \leftarrow \cdots \mathbb{Z}/p^n\mathbb{Z} \leftarrow \cdots$$

where the arrows are the natural projections.

Haar measures.
From the algebraic construction we immediately see that the space $\mathbb{Z}_p$ is compact (Tychonoff's theorem). The same is of course true for $p^n\mathbb{Z}_p$. Thus the groups $\mathbb{Q}_p$ and $\mathbb{Q}_p^\times$ are locally compact, because every element x in each of these groups has a basis of compact neighbourhood given by $x + p^n\mathbb{Z}_p$. By a general theorem we know that on each of these groups there exists an invariant Haar measure, which we will call dx and d^*x respectivelly. It is immediate that $dx = |x|d^*x$ up to a constant. We normalize dx such that the volume of the compact subgroup $\mathbb{Z}_p$ is 1.

An example from geometry.
Consider a compact Riemann surface X with its fields of meromorphic functions K. For each $x \in X$ we have an additive discrete valuation $\mathrm{ord}_x : K^\times \to \mathbb{Z}$ which gives the order of the pole or zero an element of K has at the point x. This valuation is obviously trivial on the field of constant functions $\mathbb{C} \subset K$ and it can be shown that any such valuation coincides (up to a constant) with ord_x for some $x \in X$. To every valuation ord_x we have the discrete valuation ring $\mathcal{O}_x \subset K$ consisting of those functions which are holomorphic at x. The field K is the quotient field of $\mathcal{O}_x$. Note that the rings $\mathcal{O}_x$ are not complete, because they only contain power series with positive convergence radius.

This example explains some of the terminology used when dealing with number fields and their adele rings. The valuations of K correspond $1-1$ to the points $x \in X$. This has given them the name "places". The field K algebraically encodes the analytic information of the whole object X and is called a "global field". The completions $\hat{\mathcal{O}}_x$ and their quotient fields are called "local rings" and "local fields", because they contain only information at the point x. This analogy can be made very precise using algebraic geometry. One can view the ring $\mathbb{Z}$ as a curve with points given by the prime numbers. The field $\mathbb{Q}$ is then the field of "meromorphic" functions on this curve and the valuations ord_p are an exact copy of the valuations ord_x above. A function

$q \in \mathbb{Q}$ has a zero at a point p on the curve $\mathbb{Z}$ if p divides the numerator of q and it has a pole if p divides the denominator. The "holomorphic" functions are then obviously the elements of the ring $\mathbb{Z}$ itself.

1.2 The ring of adeles

Now we want to put all completions of $\mathbb{Q}$ in one object. This object will carry the local data of the global field $\mathbb{Q}$. One could take the product over all completions, but this object would be too large. Instead, one considers what is called the **restricted product** over all completions:

$$\mathbb{A}_S := \prod_{p\in S} \mathbb{Q}_p \times \prod_{p\notin S} \mathbb{Z}_p$$

$$\mathbb{A} := {\prod_{p\leq\infty}}' \mathbb{Q}_p := \bigcup_S \mathbb{A}_S$$

where S goes over all finite sets of places with $\infty \in S$. Elements in $\mathbb{A}$ are vectors $(x_\infty, x_2, x_3, ...)$ with $x_p \in \mathbb{Q}_p$ where all but finitely many x_p lie in $\mathbb{Z}_p$ (this is analogous to allowing finitely many poles). One says that $\mathbb{A}$ is the product over all $\mathbb{Q}_p$ restricted with respect to $\mathbb{Z}_p$. The set $\mathbb{A}$ is a ring under componentwise addition and multiplication.

We topologize $\mathbb{A}$ with the restricted product topology: A topology basis on $\mathbb{A}_S$ is given by

$$\prod_{p\in S} W_p \times \prod_{p\notin S} V_p$$

where W_p is open in $\mathbb{Q}_p$, V_p is open in $\mathbb{Z}_p$ and $V_p = \mathbb{Z}_p$ for almost all p. A set $U \subset \mathbb{A}$ is open if $U \cap \mathbb{A}_S$ is open for every S. In terms of convergence this means that the sequence $a^{(n)}$ converges towards a iff

1. $a_p^{(n)}$ converges towards a_p for all p (componentwise convergence)

2. There exisits $N > 0$ s.t. for $n > N$ and all $p < \infty$ $a_p^{(n)} - a_p \in \mathbb{Z}_p$

It is quite obvious what the units in $\mathbb{A}$ are. They are the product over all $\mathbb{Q}_p^\times$ restricted with respect to $\mathbb{Z}_p^\times$.

$$\mathbb{I} := {\prod_{p\leq\infty}}' \mathbb{Q}_p^\times := \bigcup_S \left(\prod_{p\in S} \mathbb{Q}_p^\times \times \prod_{p\notin S} \mathbb{Z}_p^\times \right)$$

Again we take the restricted product topology for $\mathbb{I}$ (and not the subspace topology induced from $\mathbb{A}$).

The group $\mathbb{I} = \mathbb{A}^\times$ is called the idele group of $\mathbb{Q}$. The name idele comes from the abbreviation id.el. standing for "ideal element". The motivation for this name comes from algebraic number theory and is similar to the motivation for the name of an "ideal" of a ring. The name adele comes from "additive idele".

Observe that we have the diagonal embeddings

$$\mathbb{Q} \hookrightarrow \mathbb{A}, \quad q \mapsto (q, q, q, q, ...)$$

$$\mathbb{Q}^\times \hookrightarrow \mathbb{I}, \quad q \mapsto (q, q, q, q, ...)$$

Both of these are well defined, because for any $x \in \mathbb{Q}$, $x \neq 0$ there are only finitely many primes p which divide either the numerator or the denominator of x. For all other primes q $x \in \mathbb{Z}_q^\times$. Now we have the following important theorems

Theorem The image of $\mathbb{Q}$ in $\mathbb{A}$ is discrete and

$$\mathbb{A}/\mathbb{Q} \quad \cong \quad [0,1] \times \prod_{p<\infty} \mathbb{Z}_p$$

In particular the quotient $\mathbb{A}/\mathbb{Q}$ is compact.

Theorem The image of $\mathbb{Q}^\times$ in $\mathbb{I}$ is discrete and

$$\mathbb{I}/\mathbb{Q}^\times \quad \cong \quad (0,\infty) \times \prod_{p<\infty} \mathbb{Z}_p^\times \quad = \quad \mathbb{R}_{>0}^\times \times \prod_{p<\infty} \mathbb{Z}_p^\times$$

We want to finish this section with the remark that the adele ring $\mathbb{A}$ and the idele group $\mathbb{I}$ can be definied not only for $\mathbb{Q}$, but for an arbitrary number field. Then the finite places come from prime ideals instead of prime numbers, and there might be more than one infinite place (those come from embeddings into $\mathbb{R}$ or $\mathbb{C}$).

Haar measures.
The spaces $\mathbb{A}_S$ are locally compact, because $\prod_{p\in S} \mathbb{Q}_p$ is a finite product of locally compact spaces and $\prod_{p\notin S} \mathbb{Z}_p$ is compact by Tychonoff's theorem. Thus the space $\mathbb{A}$ is a locally compact space. The same holds for $\mathbb{I}$. Again we get measures dx and d^*x and we normalize dx so that the volume of $\prod_{p<\infty} \mathbb{Z}_p$ is 1.

1.3 Characters

In this section we will take a look at the characters of $\mathbb{Q}_p$ and $\mathbb{A}$. They will be important for us when we look at Fourier expansions of automorphic forms. The proofs of some statements will be presented here.

We begin by recalling the following

Definition: Let G be a group. The group

$$\hat{G} = \mathrm{Hom}(G, S^1)$$

is called **character group** or **dual group** of G. Note that if G is a topological group then one takes the Hom-functor in the category of topological groups. A group G is called **self-dual** if there is an isomorphism $G \cong \hat{G}$.

Let us first examine the characters of $\mathbb{Q}_p$.

Fact: The group $\mathbb{R}$ is self-dual. All its characters are given by $x \mapsto e^{2\pi ixy}$ with $y \in \mathbb{R}$.

The proof of this is very simple – any continuous function $f : \mathbb{R} \to S^1$ can be written as $e^{2\pi ih}$ with a continuous $h : \mathbb{R} \to \mathbb{R}$ and if f is a character than h has to be a homomorphism of $(\mathbb{R}, +)$, but all continuous homomorphisms of $(\mathbb{R}, +)$ are of the form $x \mapsto cx$ with $c \in \mathbb{R}$.

Now we want to define a similar "exponential" function $b \mapsto e^{2\pi ib}$ on $\mathbb{Q}_p$. We notice that its real counterpart is trivial on $\mathbb{Z}$, and since $\mathbb{Z}$ is dense in $\mathbb{Z}_p$ we are led to demand that it be trivial on $\mathbb{Z}_p$. Let now $b \in \mathbb{Q}_p$. We write

$$b = \sum_{i=-N}^{\infty} b_i p^i = \underbrace{\sum_{i=-N}^{i=-1} b_i p^i}_{b'} + \underbrace{\sum_{i=0}^{\infty} b_i p^i}_{b''}$$

and define formally

$$e^{2\pi ib} = e^{2\pi ib'} e^{2\pi ib''}$$

But $b'' \in \mathbb{Z}_p$ so the second factor is trivial, and $b' = p^{-N}a$ with $a \in \mathbb{Z}$ and the first factor makes sense. This gives us a good definition for $b \mapsto e^{2\pi ib}$ and this is obviously a character on $\mathbb{Q}_p$.

Proposition: The group $\mathbb{Q}_p$ is self-dual. All its characters are given by $b \mapsto e^{2\pi iba}$ with $a \in \mathbb{Q}_p$.

Proof: Let $\chi : \mathbb{Q}_p \to S^1$ be a character of $\mathbb{Q}_p$. Choose a small neighbourhood U of 1 in S^1. By continuity there is an m s.t. $\chi(p^m\mathbb{Z}_p) \subset U$. But $\chi(p^m\mathbb{Z}_p)$ is a subgroup of S^1 contained in U and as such must be trivial. Choose m to be the smallest integer with $\chi(p^m\mathbb{Z}_p) = 1$ and let $\chi_1 : \mathbb{Q}_p \to S^1$, $\chi_1(b) = \chi(p^m b)$. By construction χ_1 is trivial on $\mathbb{Z}_p$ and we claim that $\chi_1(b) = e^{2\pi iba}$ with $a \in \mathbb{Z}_p$. First, by the same argument used in the construction of $e^{2\pi ib}$ it is obvious that χ_1 is determined by its values on p^{-m} for all positive integers m. Let's call these values y_m. Each y_m is a p^m-th root of unity $(1 = \chi_1(1) = p^m\chi_1(p^{-m}) = y_m^{p^m})$, so $y_m = e^{2\pi i \frac{a_m}{p^m}}$. From $y_{m+1}^p = y_m$ it follows that $a_{m+1} \equiv a_m \mod p^m$. Thus $a = (a_1, a_2, ...) \in \mathbb{Z}_p$ and $\chi_1(b) = e^{2\pi i b a}$. $\square$

Next we will look at the characters of $\mathbb{A}$.

We choose for every $p < \infty$ the character $\tau_p(b) = e^{2\pi i b}$ on $\mathbb{Q}_p$ and for the infinite place $\tau_\infty(x) = e^{-2\pi i x}$. Then

$$\tau : \mathbb{A} \to S^1, \quad x \mapsto \prod_{p \leq \infty} \tau_p(x_p)$$

is a character on $\mathbb{A}$, well defined because $x_p \in \mathbb{Z}_p$ for almost all p and we claim

Proposition: The group $\mathbb{A}$ is self-dual. All its characters are given by $\tau_a(x) = \tau(ax)$ for $a \in \mathbb{A}$.

This is very easy to see: Let χ be a character on $\mathbb{A}$. For every $p \leq \infty$ the map $x_p \mapsto \chi(0, 0, ..., x_p, 0, ...)$ is a character on $\mathbb{Q}_p$ and by the self-duality of $\mathbb{Q}_p$ there is an $a_p \in \mathbb{Q}_p$ s.t. $\chi(0, 0, ..., x_p, 0, ...) = \tau_p(a_p x_p)$. Again from the fact that there are no subgroups of S^1 contained in a small neighbourhood of 1 we see that there exists a subgroup of $\mathbb{A}$ of the form $\prod_{p \in S} p^{m_p} \mathbb{Z}_p \prod_{p \notin S} \mathbb{Z}_p$ on which χ must be trivial. This implies $a_p = 1$ for $p \notin S$ i.e. the collection (a_p) is an element of $\mathbb{A}$ and $\chi(x) = \tau(ax)$.

Now recall the following

Definition: Let $H \subset G$. Then $H^\perp = \left\{ \chi \in \hat{G} \mid \chi|_H = 1 \right\}$.

We have the following exact sequence

$$1 \to H^\perp \to \hat{G} \to \hat{H} \to 1$$

Proposition: $\mathbb{Q}^\perp = \mathbb{Q}$.

Proof: $\mathbb{Q}^\perp = \left\{ \tau_a : \mathbb{A} \to S^1 \mid \tau_a|_{\mathbb{Q}} = 1 \right\} \cong \left\{ a \in \mathbb{A} \mid \tau(a\mathbb{Q}) = 1 \right\}$. The claim is $\tau(a\mathbb{Q}) = 1 \Leftrightarrow a \in \mathbb{Q}$.

Let us first see $\tau(\mathbb{Q}) = 1$. For a prime power p^m we have $\tau(\frac{1}{p^m}) = \tau_\infty(\frac{1}{p^m}) \prod \tau_l(\frac{1}{p^m})$. Since $\frac{1}{p^m} \in \mathbb{Z}_l$ for $l \neq p$ the characters τ_l are trivial and the product simplifies to

$$\tau_\infty(\frac{1}{p^m})\tau_p(\frac{1}{p^m}) = e^{-\frac{2\pi i}{p^m}} e^{\frac{2\pi i}{p^m}} = 1$$

Since τ is a character this extends to $\tau(\frac{a}{p^m}) = 1$ for all $a \in \mathbb{Z}$. But for any rational number q we have a decomposition

$$q = \frac{a_1}{p^{m_1}} + ... + \frac{a_n}{p^{m_n}}$$

(partial fractions) which proves $\tau(\mathbb{Q}) = 1$ and this immediately implies

$$a \in \mathbb{Q} \Rightarrow \tau(a\mathbb{Q}) = 1$$

or in other words $\mathbb{Q} \subset \mathbb{Q}^\perp$.

For the other direction we show $\mathbb{Q}^\perp/\mathbb{Q} = 0$. Consider $a \in \mathbb{A}$ with $\tau(a\mathbb{Q}) = 1$. In particular this means $\tau(a) = 1$. We can find a $r \in \mathbb{Q}$ s.t. all components of $a' = a + r$ for $p < \infty$ lie in $\mathbb{Z}_p$. But by our above argument $\tau(a') = 1$. Since $a_p \in \mathbb{Z}_p$ for all $p < \infty$ we get

$$1 = \tau(a') = \tau_\infty(a'_\infty)$$

This shows $a'_\infty \in \mathbb{Z}$. Letting $a'' = a' - a_\infty$ we get $a'' = (0, a''_2, a''_3, ...)$ with $a''_p \in \mathbb{Z}_p$. Now using $\tau(a''\mathbb{Q}) = 1$ we see

$$1 = \tau(a'' \frac{1}{p^k}) = \tau_p(a''_p \frac{1}{p^k})$$

for all k which implies $a''_p = 0$ for all p. Thus we have shown $\mathbb{Q}^\perp/\mathbb{Q} = \{0\}$. $\square$

Corollary:

$$\begin{aligned} \hat{\mathbb{Q}} &= \mathbb{A}/\mathbb{Q} \\ \widehat{\mathbb{A}/\mathbb{Q}} &= \mathbb{Q} \end{aligned}$$

2 The group $\mathbf{GL}_2(\mathbb{A})$

For any commutative ring with unity R we can define the group $\mathrm{GL}_2(R)$ by

$$\mathrm{GL}_2(R) = \{A \in \mathrm{Mat}(2, 2, R) | \det A \in R^\times\}$$

This gives us a definition of the group $\mathrm{GL}_2(\mathbb{A})$. These are the 2×2 matrices with adelic coefficients, whose determinant is an adelic unit. We can also consider the product of all $\mathrm{GL}_2(\mathbb{Q}_p)$ restricted with respect to $\mathrm{GL}_2(\mathbb{Z}_p)$

$$\prod_{p \le \infty}{}' \mathrm{GL}_2(\mathbb{Q}_p) = \bigcup_S \left(\prod_{p \in S} \mathrm{GL}_2(\mathbb{Q}_p) \times \prod_{p \notin S} \mathrm{GL}_2(\mathbb{Z}_p) \right)$$

This group is isomorphic to $\mathrm{GL}_2(\mathbb{A})$ via

$$\mathrm{GL}_2(\mathbb{A}) \to \prod_{p \le \infty}{}' \mathrm{GL}_2(\mathbb{Q}_p), \quad \begin{pmatrix} (a_p)_{p \le \infty} & (b_p)_{p \le \infty} \\ (c_p)_{p \le \infty} & (d_p)_{p \le \infty} \end{pmatrix} \mapsto \prod_{p \le \infty} \begin{pmatrix} a_p & b_p \\ c_p & d_p \end{pmatrix}$$

because the ring operations of $\mathbb{A}$ are defined componentwise. We will use both notations interchangably by means of this isomorphism.

The same definition can be done for any subgroup of GL_2 defined by algebraic equations, as f.e. the group SL_2 :

$$\mathrm{SL}_2(R) = \{A \in \mathrm{GL}_2(R) | \det A = 1\}$$

Again this gives us a definition of $\mathrm{SL}_2(\mathbb{A})$ and the above isomorphism is valid.

Note: In general one defines a linear algebraic group over $\mathbb{Z}$ to be a closed subgroup-scheme of the group-scheme $\mathrm{GL}_n/\mathbb{Z}$. We have just given two such schemes via their functor of points, which is a functor $\{\mathbb{Z} - \text{algebras}\} \to \{\text{groups}\}$.

Recall that in the real theory, the group $\mathrm{SL}_2(\mathbb{R})$ had the important subgroup $K_\infty = \mathrm{SO}(2)$, which was a maximal compact subgroup. This situation has an analog in the p-adic theory. A maximal comapct subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$ is given by $\mathrm{SL}_2(\mathbb{Z}_p)$. A maximal compact subgroup of $\mathrm{SL}_2(\mathbb{A})$ is now given by $SO(2) \times \prod_{p<\infty} \mathrm{SL}_2(\mathbb{Z}_p)$. In $\mathrm{GL}_2(\mathbb{Q}_p)$ a maximal compact subgroup is given by $\mathrm{GL}_2(\mathbb{Z}_p)$ and we will call it K_p.

The discrete embedding $\mathbb{Q} \hookrightarrow \mathbb{A}$ gives us a discrete embeddings $\mathrm{SL}_2(\mathbb{Q}) \hookrightarrow \mathrm{SL}_2(\mathbb{A})$ and $\mathrm{GL}_2(\mathbb{Q}) \hookrightarrow \mathrm{GL}_2(\mathbb{A})$. We have the following special case of **strong approximation**:

Proposition

$$\mathrm{SL}_2(\mathbb{Q}) \setminus \mathrm{SL}_2(\mathbb{A}) \; / \prod_{p<\infty} \mathrm{SL}_2(\mathbb{Z}_p) \quad \cong \quad \mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R})$$

Proof: An easy calculation shows that for a given prime p to any $g \in \mathrm{SL}_2(\mathbb{Q}_p)$ there is a $\gamma \in \mathrm{SL}_2(\mathbb{Q})$ s.t.

1. $\gamma g \in \mathrm{SL}_2(\mathbb{Z}_p)$
2. $\gamma \in \mathrm{SL}_2(\mathbb{Z}_l)$ for all $l \neq p$

Now given a $g \in \mathrm{SL}_2(\mathbb{A})$ there are finitely many primes $p_1, ..., p_n$ s.t. $g_{p_i} \in \mathrm{SL}_2(\mathbb{Q}_{p_i})$ and for all other primes l the component g_l lies in $\mathrm{SL}_2(\mathbb{Z}_l)$. We can now "multiply away" the poles of g at the primes $p_1, ..., p_n$ by selecting for each p_i an element γ_{p_i} with the above properties and forming the product $\gamma_{p_1} ... \gamma_{p_n} g$. Each γ_{p_i} multiplies away the pole of g at p_i by the first property and does not introduce a new pole at any other prime by the second property.

Thus we see that in the quotient $\mathrm{SL}_2(\mathbb{Q}) \setminus \mathrm{SL}_2(\mathbb{A})$ each equivalence class contains an element from $\mathrm{SL}_2(\mathbb{R}) \times \prod_{p<\infty} \mathrm{SL}_2(\mathbb{Z}_p)$. How many such elements are there in a given equivalence class? Let g be such an element at let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Then γg is another such element. On the other hand consider two elements $g_1, g_2 \in \mathrm{SL}_2(\mathbb{R}) \times \prod_{p<\infty} \mathrm{SL}_2(\mathbb{Z}_p)$ from the same equivalence class mod $\mathrm{SL}_2(\mathbb{Q})$. Let $\gamma \in \mathrm{SL}_2(\mathbb{Q})$ be s.t. $\gamma g_1 = g_2$. Then $\gamma = g_1^{-1} g_2$ and we see that $\gamma_p \in \mathbb{Z}_p$ for all p. This means that no prime p divides the denominators of the entries of γ which that they are integers. We see

$$\mathrm{SL}_2(\mathbb{Q}) \setminus \mathrm{SL}_2(\mathbb{A}) \quad \cong \quad \mathrm{SL}_2(\mathbb{Z}) \setminus \left(\mathrm{SL}_2(\mathbb{R}) \times \prod_{p<\infty} \mathrm{SL}_2(\mathbb{Z}_p) \right)$$

$\square$

Corollary

$$\mathrm{GL}_2(\mathbb{Q}) \setminus \mathrm{GL}_2(\mathbb{A}) \ / \prod_{p<\infty} \mathrm{GL}_2(\mathbb{Z}_p) \quad \cong \quad \mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{GL}_2^+(\mathbb{R})$$

Proof: Let $g \in \mathrm{GL}_2(\mathbb{A})$. Then $\det(g) \in \mathbb{I}$ and we can factorize $\det(g) = ru$ with $r \in \mathbb{Q}^\times$ and $u \in \mathbb{R}_{>0}^\times \times \prod_{p<\infty} \mathbb{Z}_p$. This implies that g can be written as

$$\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} g_1 \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$$

with $g_1 \in \mathrm{SL}_2(\mathbb{A})$ and the above proposition can be applied.

$\square$

Corollary Let $Z \subset \mathrm{GL}_2$ be the center of GL_2:

$$Z(R) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in R^\times \right\}$$

Then

$$Z(\mathbb{A})\mathrm{GL}_2(\mathbb{Q}) \setminus \mathrm{GL}_2(\mathbb{A}) \ / \prod_{p<\infty} \mathrm{GL}_2(\mathbb{Z}_p) \quad \cong \quad \mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R})$$

$$Z(\mathbb{A})\mathrm{GL}_2(\mathbb{Q}) \setminus \mathrm{GL}_2(\mathbb{A}) \ / SO(2) \prod_{p<\infty} \mathrm{GL}_2(\mathbb{Z}_p) \quad \cong \quad \mathrm{SL}_2(\mathbb{Z}) \setminus \mathbb{H}$$

Remark: Instead of taking $\prod_{p<\infty} \mathrm{GL}_2(\mathbb{Z}_p)$ one could take $\prod_{p<\infty} K_p^N$ where

$$K_p^N = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) \middle| \ c \equiv 0 \mod N \right\}$$

If $p \nmid N$ then the condition is vacuous (because N is a unit in $\mathbb{Z}_p$) and $K_p^N = \mathrm{GL}_2(\mathbb{Z}_p)$. If $p^{v(N)} || N$ then

$$K_p^N = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) \middle| \ c \in p^{v(N)} \mathbb{Z}_p \right\}$$

All of the above statements hold for $\prod K_p$ replaced by $\prod K_p^N$ with $\mathrm{SL}_2(\mathbb{Z})$ replaced by

$$\Gamma_0(N) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in \mathrm{SL}_2(\mathbb{Z}) \middle| \, c \equiv 0 \mod N \right\}$$

3 Automorphic forms as functions on $\mathbf{GL}_2(\mathbb{A})$

In this section we will describe the classical automorphic forms as functions on $\mathrm{GL}_2(\mathbb{A})$. This new notion has two very important advantages – it allows generalizations which include non-holomorphic cusp-forms or Maass waveforms, and it is the starting point for the use of representation theory in the study of automorphic forms. For this section we follow [Ge] Chap. 3 very closely.

Let G denote the group GL_2.

3.1 Characterization and definition of automorphic forms on $\mathbf{GL}_2(\mathbb{A})$

For $f \in S_k(N)$ we define the function ϕ_f on $G(\mathbb{A})$ by considering the decomposition $G(\mathbb{A}) = G(\mathbb{Q})G(\mathbb{R})^+ \prod K_p^N$ and setting

$$\phi_f(\gamma g_\infty k) = f(g_\infty \cdot i) j(g_\infty, i)^{-k}$$

We have already seen the same construction when we transfered the classical cusp forms to functions on $\mathrm{SL}_2(\mathbb{R})$. Indeed, if we regard the above function as a function of g_∞ alone then it coincides with the previous definition. In particular this function of g_∞ is invariant under $\Gamma_0(N)$ and since

$$G(\mathbb{Q}) \cap G(\mathbb{R})^+ \prod K_p^N = \Gamma_0(N)$$

it follows that the above function is well defined. Note further that although the definition of ϕ_f involves only g_∞ the function actually depends on all components of g — if we were to take a $g = \gamma g_\infty k$ and change the component g_p for any $p < \infty$ so that it lies outside of K_p^N then there is a $\gamma_1 \in G(\mathbb{Q})$ s.t. all p-components of $\gamma_1 g$ lie in K_p^N. But the ∞-component of this element is now different from g_∞ and thus the value of ϕ_f is also different.

Proposition The map

$$S_k(N) \to \{\phi : G(\mathbb{A}) \to \mathbb{C}\}, \quad f \mapsto \phi_f$$

is an isomorphism into the space of functions ϕ on $G(\mathbb{A})$ satisfying the following conditions

1. $\phi(\gamma g) = \phi(g)$ for all $\gamma \in G(\mathbb{Q})$
2. $\phi(gk) = \phi(g)$ for all $k \in \prod K_p^N$
3. $\phi\left(g\begin{pmatrix} \cos(\vartheta) & -\sin(\vartheta) \\ \sin(\vartheta) & \cos(\vartheta) \end{pmatrix}\right) = e^{-ik\vartheta}\phi(g)$
4. Viewed as a function of $G(\mathbb{R})^+$ alone, ϕ satisfies the differential equation
$$\triangle\phi = -\frac{k}{2}(\frac{k}{2} - 1)\phi$$
5. $\phi(zg) = \phi(g)$ for all $z \in Z(\mathbb{A})$
6. For every $c > 0$ and $\omega \subset G(\mathbb{A})$ compact there exist constants C and N s.t.
$$\phi(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g) \leq C|a|^N$$
for all $g \in \omega$ and $a \in \mathbb{I}$ with $|a| > c$.
7. $$\int_{\mathbb{Q}\backslash\mathbb{A}} \phi(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g)dx = 0$$
for almost every g.

Proof: We have already seen a very similar statement in Jörn's talk. It is obvious that ϕ_f satisfies $1, 2, 3, 5$ and we know the calculation which shows 4. Furthermore we know that a ϕ which satisfies $1 - 5$ defines a holomorphic function on $\Gamma_0(N) \setminus \mathbb{H}$.

Condition 6 is the equivalent of the regularity of f at the cusps. We will sketch how it translates into the similar condition

$$\phi_\infty(x + iy, \vartheta) < C|y|^N$$

for $\phi_\infty : G(\mathbb{R}) \to \mathbb{C}$ which was defined in Jörn's talk. The latter is equivalent to
$$\phi_\infty(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c & -s \\ s & c \end{pmatrix}) < C|y|^N$$
by definition of the coordinates x, y, ϑ and the invariance of ϕ_∞ under the center of $G(\mathbb{R})$.

We fix g for convenience (it varies compactly anyway) and decompose

$$g = \gamma g_\infty k_2 k_3 ...$$

with $\gamma \in G(\mathbb{Q})$, $k_p \in G(\mathbb{Z}_p)$. Then $\phi_f(g) = \phi_{f,\infty}(g_\infty)$. Let $a \in \mathbb{I}$, $a = a_\infty a_2 a_3 ...$, $\underline{a} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$. We want to estimate the value of ϕ_f at the point

$\underline{a}g$. To do this we need to figure out a decomposition $\underline{a}g = \gamma' g'_\infty k'_2 k'_3 \ldots$. There are finitely many p with $\mathrm{ord}_p(a_p) \neq 0$. Let $q_p = p^{-\mathrm{ord}_p(a_p)}$ for all p. Then $q_p = 1$ for almost all p and we can form $q = \prod q_p$. This is a rational number and $(qa)_p \in \mathbb{Z}_p^\times$ for all p (we have multiplied away the poles of a). Thus

$$\begin{pmatrix} qa & 0 \\ 0 & 1 \end{pmatrix} g = \gamma \begin{pmatrix} qa_\infty & 0 \\ 0 & 1 \end{pmatrix} g_\infty k'_2 k'_3 \ldots$$

with the same γ from the decomposition of g. But by definition $|q|_\infty = \prod |q_p|_\infty = \prod |a_p|_p$ and so $|qa_\infty|_\infty = |a|$. Thus

$$\phi_f(\underline{a}g) = \phi_{f,\infty}(\begin{pmatrix} qa_\infty & 0 \\ 0 & 1 \end{pmatrix} g_\infty) < C|qa_\infty|_\infty^N = C|a|^N$$

Condition 7 is equivalent to the cuspidality condition for f. We will show this when we compute the Fourier coefficients of ϕ_f.

$\square$ (up to 7).

Proposition The space of functions satisfying the conditions $1-7$ is a subspace of $L^2(Z(\mathbb{A})G(\mathbb{Q}) \setminus G(\mathbb{A}))$.

Proof: This is immediately clear from the fact that

$$Z(\mathbb{A})G(\mathbb{Q}) \setminus G(\mathbb{A}) \;/SO(2) \prod_{p<\infty} K_p^N \quad \cong \quad \Gamma_0(N) \setminus \mathbb{H}$$

which implies

$$\begin{aligned} \int_{Z(\mathbb{A})G(\mathbb{Q})\setminus G(\mathbb{A})} |\phi_f(g)|^2 dg &= C \int_{Z(\mathbb{A})G(\mathbb{Q})\setminus G(\mathbb{A})/K} |\phi_f(g)|^2 dg \\ &= C \int_{\Gamma_0(N)\setminus\mathbb{H}} |f(z)|^2 y^k \frac{dxdy}{y^2} < \infty \end{aligned}$$

Here C refers to a general constant.

$\square$

Now we generalize the conditions $1-7$ to form the new definition of an automorphic form

Definition: An automorphic form on GL_2 is a function $\phi : \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$ satisfying

1. $\phi(\gamma g) = \phi(g)$ for all $\gamma \in G(\mathbb{Q})$
2. $\phi(zg) = \phi(g)$ for all $z \in Z(\mathbb{A})$
3. ϕ is right K-finite

4. As a function of $G(\mathbb{R})$ alone ϕ is $\mathfrak{z}$-finite (and smooth)

5. ϕ is slowly increasing in the sense of condition 6 above

If in addition ϕ satisfies the cuspidal condition (7 above) the ϕ is called a cusp form. We shall call the space of these functions A_0.

Remark: One can use reduction theory to show that any cusp form (not only the ones coming from classical cusp forms on $\mathbb{H}$) is an element of $L^2(Z(\mathbb{A})G(\mathbb{Q}) \setminus G(\mathbb{A}))$. If we define $L_0^2(Z(\mathbb{A})G(\mathbb{Q}) \setminus G(\mathbb{A}))$ to be the subspace of L^2 of functions satisfying the cuspidality condition, then the space A_0 is the dense subspace of it consisting of K-finite and $\mathfrak{z}$-finite vectors. Thus we shall use the term cusp form also for elements of L_0^2.

3.2 The Fourier coefficients of an automorphic form on $\mathbf{GL}_2(\mathbb{A})$

First we recall the following

Fact

- The group $\mathbb{R}$ is self-dual. All its characters are given by

$$\tau_{\infty,y} : x \mapsto e^{2\pi i x y} \quad \text{for} \quad y \in \mathbb{R}$$

- The group $\mathbb{Q}_p$ is self-dual. All its characters are given by

$$\tau_{p,b} : x \mapsto e^{2\pi i} \quad ... \text{ for } \quad b \in \mathbb{Q}_p$$

- The group $\mathbb{A}$ is self-dual. If we fix the characters $\tau_\infty = \tau_{\infty,-1}$ and $\tau_p = \tau_{p,1}$ then

$$\tau(x) = \prod_{p \leq \infty} \tau_p(x_p)$$

  is a character on $\mathbb{A}$ and all characters on $\mathbb{A}$ are given by $\tau_a(x) = \tau(ax)$ for $a \in \mathbb{A}$

- The dual group of $\mathbb{Q} \setminus \mathbb{A}$ is $\mathbb{Q}$. The characters on $\mathbb{Q} \setminus \mathbb{A}$ are given by τ_ξ with $\xi \in \mathbb{Q}$.

Now given a cusp form ϕ the function

$$x \mapsto \phi(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g)$$

is a square-integrable function on the compact abelian group $\mathbb{Q}\backslash\mathbb{A}$ for almost all g and we have the usual Fourier expansion (or Peter-Weyl theorem)

$$\phi(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g) = \sum_{\xi\in\mathbb{Q}} \phi_\xi(g)\tau(\xi x)$$

where $\phi_\xi(g)$ is the ξ-th Fourier coefficient of ϕ given by

$$\phi_\xi(g) = \int_{\mathbb{Q}\backslash\mathbb{A}} \phi(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g)\overline{\tau(\xi x)}dx$$

It is now obvious that the cuspidal condition means exactly $\phi_0(g) = 0$. Furthermore, if $\phi = \phi_f$ for $f \in S_k(N)$ then the Fourier expansion of ϕ contains the Fourier expansions of f at all cusps.

We will show this for the case $N = 1$

Lemma: Let $f \in S_k(1)$, $\phi = \phi_f$. For each $y > 0$

$$\phi_\xi(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}) = \begin{cases} a_n e^{2\pi n y} & \text{if} \quad \xi = n \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$\phi(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}) = \sum_{\xi\in\mathbb{Q}} \phi_\xi(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix})\tau(\xi x) = \sum_n a_n e^{2\pi i n z} = f(z)$$

Remark: This lemma tells us in particular $\phi_0(y) = 0 \Leftrightarrow a_0 = 0$ which completes the proof of the proposition from the previous subsection.

Proof: Suppose first $\xi \notin \mathbb{Z}$. Then for some prime p and integer $m > 0$ $\xi = \alpha p^{-m}$ with $\alpha \in \mathbb{Q}$ relatively prime to p. Since $y > 0$ is real let us denote it y_∞. Now

$$\phi_\xi(\begin{pmatrix} y_\infty & 0 \\ 0 & 1 \end{pmatrix}) = \int_{\mathbb{Q}\backslash\mathbb{A}} \phi(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_\infty & 0 \\ 0 & 1 \end{pmatrix})\overline{\tau(\xi x)}dx$$

We now that $\mathbb{Q} \setminus \mathbb{A} \cong [0,1] \times \prod K_p$ and we split $x = x_\infty k$ with $x_\infty \in [0,1]$ and $k \in \prod K_p$. The function ϕ is right invariant under $\prod K_p$ so we get

$$\phi_\xi(\begin{pmatrix} y_\infty & 0 \\ 0 & 1 \end{pmatrix}) = \int_0^1 \phi(\begin{pmatrix} 1 & x_\infty \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_\infty & 0 \\ 0 & 1 \end{pmatrix})\overline{\tau(\xi x_\infty)}dx$$

Let $t = (0, 0, ..., p^{m-1}, 0, 0...)$ where the p^{m-1} is at the p-th place. Then

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in \prod K_p$$

and

$$\begin{aligned}
& \int_0^1 \phi(\begin{pmatrix} 1 & x_\infty \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_\infty & 0 \\ 0 & 1 \end{pmatrix})\overline{\tau(\xi x_\infty)}dx \\
= & \int_0^1 \phi(\begin{pmatrix} 1 & x_\infty \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_\infty & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix})\overline{\tau(\xi x_\infty)}dx \\
= & \int_0^1 \phi(\begin{pmatrix} 1 & x_\infty + t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_\infty & 0 \\ 0 & 1 \end{pmatrix})\overline{\tau(\xi x_\infty)}dx \\
= & \int_0^1 \phi(\begin{pmatrix} 1 & x_\infty \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_\infty & 0 \\ 0 & 1 \end{pmatrix})\overline{\tau(\xi(x_\infty - t))}dx \\
= & \tau(\xi t)\int_0^1 \phi(\begin{pmatrix} 1 & x_\infty \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_\infty & 0 \\ 0 & 1 \end{pmatrix})\overline{\tau(\xi x_\infty)}dx \\
= & \tau(\xi t)\phi_\xi(\begin{pmatrix} y_\infty & 0 \\ 0 & 1 \end{pmatrix})
\end{aligned}$$

But $\tau(\xi t) = \tau((0, 0, ..., \alpha p^{-1}, 0, ...)) \neq 1$ by our choice of τ. Thus

$$\phi_\xi(\begin{pmatrix} y_\infty & 0 \\ 0 & 1 \end{pmatrix}) = 0$$

Now suppose $\xi = n \in \mathbb{Z}$. Then

$$\begin{aligned}
\phi_\xi(\begin{pmatrix} y_\infty & 0 \\ 0 & 1 \end{pmatrix}) &= \int_0^1 \phi(\begin{pmatrix} y_\infty & x_\infty \\ 0 & 1 \end{pmatrix})\overline{\tau(n x_\infty)}dx \\
&= \int_0^1 f(x+iy)e^{-2\pi i n x} = a_n e^{-2\pi n y}
\end{aligned}$$

$\square$

Remark: Suppose the group Γ had another cusp $s \neq \infty$ and let $\sigma \in \mathrm{SL}_2(\mathbb{Z})$ map ∞ to s. In Jörn's talk we saw that the Fourier coefficients of an automorphic form f for Γ around the cusp s are given by the integrals

$$\int_0^1 \phi_{f,\infty}(\sigma \begin{pmatrix} 1 & hx \\ 0 & 1 \end{pmatrix} g)e^{-2\pi i x}dx$$

where $h = h_\infty$ is the width of the cusp, $\phi_{f,\infty}$ is the function on $\mathrm{SL}_2(\mathbb{R})$ defined by f and $g = g_\infty \in \mathrm{SL}_2(\mathbb{R})$. But if we set $\gamma = \sigma \begin{pmatrix} h_\infty & 0 \\ 0 & 1 \end{pmatrix} \in G(\mathbb{Q})$, $g'_\infty = \begin{pmatrix} h_\infty^{-1} & 0 \\ 0 & 1 \end{pmatrix} g_\infty \in G(\mathbb{R})$ and $k_p = \sigma \in G(\mathbb{Z}_p)$ then

$$\begin{aligned}
\int_0^1 \phi_{f,\infty}(\sigma \begin{pmatrix} 1 & hx \\ 0 & 1 \end{pmatrix} g)dx &= \int_0^1 \phi_f(\sigma \begin{pmatrix} 1 & h_\infty x_\infty \\ 0 & 1 \end{pmatrix} g_\infty)dx_\infty \\
&= \int_0^1 \phi_f(\gamma \begin{pmatrix} 1 & x_\infty \\ 0 & 1 \end{pmatrix} g'_\infty k_2 k_3 ...)dx_\infty \\
&= \phi_\xi(g'_\infty k)
\end{aligned}$$

which shows us that the generalized Foruier expansion indeed contains the Fourier expansions of f for *all* cusps.

4 The Hecke operators in the adelic setting

In the previous section we considered the injection

$$S_k(N) \hookrightarrow L_0^2(Z(\mathbb{A})G(\mathbb{Q}) \setminus G(\mathbb{A})/\prod K_p^N)$$

Let p be a prime not dividing N. Then $K_p^N = K_p$ and we can consider the operator

$$\tilde{T}(p)\phi(g) = \int\limits_{K_p\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}K_p} \phi(gh)dh$$

which is the convolution of ϕ with the characteristic function of the double coset $K_p\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}K_p$. It is very easy to see that this operator leaves the image of $S_k(N)$ invariant and we have the following proposition

Lemma:

$$K_p\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}K_p = \bigcup_{t=0}^{p-1}\begin{pmatrix} p & -t \\ 0 & 1 \end{pmatrix}K_p \cup \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}K_p$$

and the union is disjoint.

Proof: First we see that the union is disjoint. Suppose it were not, say

$$\begin{pmatrix} p & -t_1 \\ 0 & 1 \end{pmatrix}k = \begin{pmatrix} p & -t_2 \\ 0 & 1 \end{pmatrix}k'$$

for $k, k' \in K_p$ and $t_1 \neq t_2$. But then

$$\begin{pmatrix} 1 & p^{-1}(t_2 - t_1) \\ 0 & 1 \end{pmatrix} = k'k^{-1} \in K_p$$

which is an obvious contradicition. The same argument shows that the last part of the union is disjoint from the rest.

Next we see that the union is a subset of the double coset, because

$$\begin{aligned}
\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}K_p &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}K_p \\
\begin{pmatrix} p & -t \\ 0 & 1 \end{pmatrix}K_p &= \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}K_p
\end{aligned}$$

The last step is to see that every matrix $k\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}k''$ falls in one of the $p+1$ right K_p-cosets. We can obviously forget k'' and we try to find a k' with

$$k\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p & -t \\ 0 & 1 \end{pmatrix}k'$$

which would put our matrix in one of the first p right K_p-cosets.

Let $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$\begin{pmatrix} p^{-1} & p^{-1}t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+tc & p^{-1}b+p^{-1}td \\ pc & d \end{pmatrix} \stackrel{!}{=} k'$$

The determinant of the candidate for k' is obviously $ad-bc \in \mathbb{Z}_p^\times$ which is ok. Now we need that the entries come from $\mathbb{Z}_p$. We need to distinguish four cases. If $p|b \wedge p|d$ then we are ok. If $p|b \wedge p \nmid d$ then we can choose $t=0$. If $p \nmid b$ and $p \nmid d$ then both b and d are units in $\mathbb{Z}_p$ and we can choose $t = -\frac{b}{d}$ which makes the problematic entry vanish. The problem is only if $p \nmid b \wedge p|d$. In this case we have another candidate for k', namely

$$\begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} pa & b \\ c & p^{-1}d \end{pmatrix}$$

which works out and puts our original matrix in the last coset.

$\square$

We will give names for the matrices in the above decomposition to refer to them more easily

$$\xi_b = \begin{pmatrix} p & -b \\ 0 & 1 \end{pmatrix} \text{ for } b = 0, ..., p-1 \qquad \xi_b = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \text{ for } b = p$$

Proposition: Let $f \in S_k(N)$. Then

$$p^{\frac{k}{2}-1}\tilde{T}(p)\phi_f = \phi_{T(p)f}$$

Proof: By the above decomposition and the right-invariance of $\phi = \phi_f$ under K_p we get

$$\begin{aligned}
\tilde{T}(p)\phi(g) &= \int_{K_p\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)K_p} \phi(gh_p)dh_p \\
&= \sum_{b=0}^{p-1}\int_{\left(\begin{smallmatrix} p & -b \\ 0 & 1 \end{smallmatrix}\right)K_p} \phi(gh_p)dh_p + \int_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)K_p} \phi(gh_p)dh_p \\
&= \sum_{b=0}^{p}\phi(g[\xi_b]_p)
\end{aligned}$$

where

$$[\xi_b]_p := \left(1, 1, ..., \underbrace{\xi_b}_{p\text{-th place}}, 1, 1, ... \right) \in \prod_{l \neq p} K_p^N \times \mathrm{GL}_2(\mathbb{Q}_p)$$

In order to evaluate the above sum we need to figure out the infinite component of $g[\xi_b]_p$ in the decomposition $G(\mathbb{Q})G(\mathbb{R})^+K^N$. Let's look at the case $\xi_b = \left(\begin{smallmatrix} p & -b \\ 0 & 1 \end{smallmatrix} \right)$. Decompose $g = \gamma g_\infty k$. Then $g \left(\begin{smallmatrix} p & -b \\ 0 & 1 \end{smallmatrix} \right)_p = \gamma g_\infty y$ where $y_p = k_p \left(\begin{smallmatrix} p & -b \\ 0 & 1 \end{smallmatrix} \right)$ and $y_l = k_l$ for $l \neq p$. But $\gamma g_\infty y$ is not a decomposition we can use because $y_p \notin G(\mathbb{Z}_p)$. Obviously y_p is an element of the double coset and thus we can find a $k'_p \in G(\mathbb{Z}_p)$ s.t. either $y_p = \left(\begin{smallmatrix} p & -t \\ 0 & 1 \end{smallmatrix} \right) k'_p$ for a suitable t or $y_p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right) k'_p$. We can't directly see which of these will be the case, but what we can directly see is that the two matrices

$$y_{1,p} = k_p \begin{pmatrix} p & -b_1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad y_{2,p} = k_p \begin{pmatrix} p & -b_2 \\ 0 & 1 \end{pmatrix}$$

where we have fixed k_p and $b_1 \neq b_2$ cannot fall in the same right K_p-coset. Assuming this were the case, say

$$y_{1,p} = \begin{pmatrix} p & -t \\ 0 & 1 \end{pmatrix} k_{1,p} \qquad y_{2,p} = \begin{pmatrix} p & -t \\ 0 & 1 \end{pmatrix} k_{2,p}$$

we come to

$$\begin{pmatrix} 1 & p^{-1}(b_2 - b_1) \\ 0 & 1 \end{pmatrix} = y_{2,p}^{-1} y_{1,p} = k_{2,p}^{-1} k_{1,p} \in G(\mathbb{Z}_p)$$

which is a contradiction. We can treat the case $\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right)$ in the same way and arrive at the following statement:

All the matrices $k_p[\xi_b]_p$ can be written as $[\xi_t]_p k'_p$ and for two different $b_1 \neq b_2$ we get two different $t_1 \neq t_2$. Thus we get

$$\sum_{b=0}^{p} \phi(\gamma g_\infty k[\xi_b]_p) = \sum_{t=0}^{p} \phi(\gamma g_\infty [\xi_t]_p k')$$

Now let

$$\gamma'_t = \gamma \xi_t, \quad g'_\infty = \xi_t^{-1} g_\infty, \quad k''_p = k'_p, \quad k''_l = [\xi_t]_l^{-1} k'_l \text{ for all } l \neq p$$

Then $\gamma g_\infty [\xi_t]_p k' = \gamma' g'_\infty k''$ and $k'' \in K^N$ and setting $z = g_\infty i$ we get

$$\phi(\gamma g_\infty [\xi_t]_p k') = \phi(\gamma' g'_\infty k'') = f(g'_\infty \cdot i) j(g'_\infty, i) = f(\xi_t^{-1} z) j(\xi_t, z)^{-k} j(g_\infty, i)^{-k}$$

For $\xi_t = \begin{pmatrix} p & -t \\ 0 & 1 \end{pmatrix}$ we get

$$f(\xi_t^{-1} z) = f(\frac{z + t}{p}), \quad j(\xi_t^{-1}, z)^{-k} = p^{-\frac{k}{2}}$$

and for $\xi_t = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$

$$f(\xi_t^{-1} z) = f(pz), \quad j(\xi_t^{-1}, z)^{-k} = p^{\frac{k}{2}}$$

Consequently

$$\begin{aligned} p^{\frac{k}{2}-1}\tilde{T}(p)\phi(g) &= p^{-1}\sum_{t=0}^{p-1} f(\frac{z+t}{p}) j(g_\infty, i)^{-k} + p^{k-1} f(pz) j(g_\infty, i)^{-k} \\ &= p^{k-1} \sum_{\substack{a>0 \\ ad=p}} \sum_{b=0}^{d-1} f(\frac{az+b}{d}) d^{-k} j(g_\infty, i)^{-k} \\ &= T(p) f(g_\infty \cdot i) j(g_\infty, i)^{-k} \\ &= \phi_{T(p)f}(g) \end{aligned}$$

□

5 An outlook on representation theory

In this last section we want to give a general overview on how representation theory enters the field of automorphic forms. Representation theory is absolutely central in the modern treatment of automorphic forms and in this way it is introduced into number theory in a very conceptual way by means of the Langlands program.

Because of the time constraints of this talk we will present no proofs at all.

5.1 Old and new forms

We begin by recalling the classical notion of *old* and *new* forms. Choose an integer $N > 0$. An $f \in S_k(N)$ will be called an *old* form, if for m a proper divisor of N and d a divisor of $\frac{N}{m}$ there exists a $g \in S_k(m)$ with $f(z) = g(dz)$. It is called *old* because it comes from forms for $\Gamma_0(m)$ and in this sense is not *new* for the group $\Gamma_0(N)$. Logically, a form $f \in S_k(N)$ which does not satisfy the above condition will be called a *new* form.

We will denote the space of new forms by $S_k^+(N)$ and the space of old forms $S_k^-(N)$. If $f(z) = g(dz)$ is an old form, then

$$f(z) = g\Big|\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}_k$$

and for $p \nmid N$

$$\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}_k \cdot T(p) = T(p) \cdot \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}_k$$

For this reason the Hecke algebra (for $p \nmid N$) preserves the space $S_k^-(N)$ and also the space $S_k^+(N)$ which is its orthogonal complement with respect to the Petrson scalar product and we can choose a basis for $S_k(N)$ of simultaneous eigenfunctions for this algebra such that part of the basis elements are old forms and the others are new forms.

Observe further that every old form comes from a new form, i.e. in the equality $f(z) = g(dz)$ the form g can be chosen to be a new form (this is obvious). We will call the set of forms $f(z) = g(dz)$ where g is a fixed new form on $S_k(m)$ and d varies among the divisors of $\frac{N}{m}$ an *old class*.

Why is the distinction between old and new forms important? It is because of the fact that the space $S_k^+(N)$ has a basis of eigenfuctions for **all** Hecke operators, not only for those with $p \nmid N$. Thus a new form is completely characterized (up to a scalar multiple) by its Hecke eigenvalues. More precisely we have the following

Theorem

1. The space $S_k(N)$ has a basis consisting of old forms and new forms
2. Each new form in this basis is an eigenfunction for all Hecke operators. The eigenvalues for $T(p)$ with $p|N$ are independend of the new form. They are 0 if $p^2|N$ and $\pm p^{\frac{k}{2}-1}$ if $p||N$.
3. Two elements of the given basis share the same eigenvalues for all $T(p)$ with $p \nmid N$ if and only if they are both old forms and come from the same new form (i.e. lie in the same old class).

5.2 The representations generated by automorphic forms

We have already discussed the injection

$$S_k(N) \hookrightarrow L_0^2(Z(\mathbb{A})G(\mathbb{Q}) \setminus G(\mathbb{A})/K_0^N)$$

It can be shown that the space on the right decomposes as a discrete direct sum of irreducible representations with finite multiplicities. The arguments are essentially the same as the one used to show that $L_0^2(\Gamma_0(N) \setminus \mathbb{H})$ decomposes as a discrete direct sum. From now on we will use the short notation L_0^2 for $L_0^2(Z(\mathbb{A})G(\mathbb{Q}) \setminus G(\mathbb{A})/K_0^N)$.

Theorem (Multiplicity one)
Every irreducible subrepresentation of L_0^2 has multiplicity one.

To a function $f \in S_k(N)$ we can assign a representation of $G(\mathbb{A})$ inside L_0^2 by taking the linear span of $\{g \cdot \phi_f | \ g \in G(\mathbb{A})\}$. This is a unitary representation of $G(\mathbb{A})$ and we call it π_f. We have the following theorem which describes the relation between forms in $S_k(N)$ and irreducible unitary representations of $G(\mathbb{A})$.

Theorem

1. Let $f \in S_k(N)$ be an eigenfunction for all $T(p)$ with $p \nmid N$. Then π_f is irreducible.

2. Let π be an irreducible subrepresentation of L_0^2. Define $\{f_\pi\}$ to be the set of functions $f_\pi \in S_k(N)$ sharing the same eigenvalues for all $T(p)$ with $p \nmid N$ and the conductor of π and such that ϕ_{f_π} lie in the space of π. Then the correspondence

$$\{f\} \leftrightarrow \pi_f$$

   is one to one

3. The map $f \mapsto \pi_f$ is one-to-one on $S_k^+(N)$ and it is not one-to-one on $S_k^-(N)$.

This theorem tells us that to study automorphic forms we need to understand the irreducible $G(\mathbb{A})$-submodules of $L_0^2(Z(\mathbb{A})G(\mathbb{Q}) \setminus G(\mathbb{A})/K_0^N)$. An essential part in understanding such irreducible representations of $G(\mathbb{A})$ lies in understanding the irreducible unitary representations of the local groups $G(\mathbb{R})$ and $G(\mathbb{Q}_p)$ as is shown by the following

Theorem
Every irreducible subrepresentation π of L_0^2 factorizes as a restricted tensor product of local irreducible unitary representations

$$\pi = \bigotimes_{p \leq \infty}{}' \pi_p$$

Note: We deliberately omit the technical notion of admissibility for the purposes of this overview.

Remark: We want to give a short explanation of what a *restricted tensor product* means. The idea is the same as with the restricted direct product we already saw. Let $\{\pi_p\}$ be a collection of local irreducible unitary representations. We assume that for almost all p the representation π_p contans a vector ξ_p^0 which is fixed by K_p. Such a representation is called *spherical* and the vector ξ_p^0 is called *spherical vector*. Fix once and for all a spherical unit vector ξ_p^0 for almost all p. Then the space of $\otimes'_p \pi_p$ is generated by the elements $\otimes'_p \xi_p$ for which $\xi_p = \xi_p^0$ for almost all p. The action of the group $G(\mathbb{A})$ on this space is given componentwise. Note that for a $g = (g_p) \in G(\mathbb{A})$ almost all components g_p lie in K_p. Thus the action of such a g on $\otimes' \xi_p$ changes only finitely many components.

Given the above theorem one is lead to study irreducible unitary representations of $G(\mathbb{R})$ and $G(\mathbb{Q}_p)$. We have already discussed the case of $\mathrm{SL}_2(\mathbb{R})$. The case of $\mathrm{GL}_2(\mathbb{R})$ is very similar. The theory for $G(\mathbb{Q}_p)$ on the other hand is quite different. One similarity is that one can define induced representations from a parabolic subgroup and show a submodule theorem similar to the real case. Yet this does not exhaust all possible representations. What is left are the so called "supercuspidal" representations which do not occur in the real theory. Another difference is that the Lie-algebra plays no role in classifying the representations. One uses the Hecke group algebra $\mathcal{H}(G)$ of locally constant compactly supported functions on $G(\mathbb{Q}_p)$. For the classification of irreducible admissible representations of $G(\mathbb{Q}_p)$ and unitarizability one can refer to [Go].

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