THE WEYL CHARACTER FORMULA

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ABSTRACT. Let U be a compact connected semisimple Lie group and $T \subset U$ be its maximal torus. Further let W the Weyl group of U, i.e.,

$$W = \text{Normalizer of } T \text{ in } U/\text{Centralizer of } T \text{ in } U.$$

Let $R(U)$ be the representation ring of U and Λ be the weight lattice. Let $\mathbb{Z}[\Lambda]$ be the group algebra of the group Λ with coefficients in $\mathbb{Z}$; by definition $\mathbb{Z}[\Lambda]$ has a basis $\{e^\lambda \mid \lambda \in \Lambda\}$, such that $e^\lambda \cdot e^{\lambda'} = e^{\lambda+\lambda'}$. Define a *character homomorphism*

$$\chi\colon R(U) \to \mathbb{Z}[\Lambda], \quad \chi_V = \chi_\pi = \sum \dim V_\lambda e^\lambda,$$

where $V_\lambda = \{v \in V \mid \pi(t)v = e^\lambda(t)v \quad \forall\, t \in T\} \neq \{0\}$ is the corresponding weight space of (π, V) for to the weight λ.

Theorem (WEYL FORMULA (1925))**.** *Let V be a finite dimensional irreducible representation of U and χ_V its character. Then*

$$\chi_V\restriction_T = \frac{1}{\prod_{\alpha\in\Phi^+} e^{\alpha/2} - e^{-\alpha/2}} \sum_{w\in W} \operatorname{sign}(w) e^{w(\lambda+\rho)}.$$

1. HOLOMORPHIC LEFSCHETZ FORMULA

Let X be a compact complex manifold of dimension $\dim_{\mathbb{C}} X = n$. The complex cotangential bundle splits into a direct sum of holomorphic and antiholomorphic cotangential bundle

$$T^*X \otimes \mathbb{C} = (T^{1,0}X)^* \oplus (T^{0,1}X)^*. \tag{1}$$

Corresponding to this decomposition the bundle of the complexified de Rham complex decompose into the tensor product $\Lambda^*(T^*X \otimes \mathbb{C}) = \Lambda^*(T^{1,0}X)^* \otimes \Lambda^*(T^{0,1}X)^*$, so that

$$\Lambda^r T^*X \otimes \mathbb{C} = \bigoplus_{p+q=r} \Lambda^p(T^{1,0}X)^* \otimes \Lambda^q(T^{0,1}X)^* =: \bigoplus_{p+q=r} \Lambda^{p,q}. \tag{2}$$

The exterior derivative $\mathrm{d}\colon \Lambda^r(X) \to \Lambda^r(X)$ decompose correspondingly to (1) into a direct sum $\partial + \overline{\partial}$, where

$$\partial\colon \Lambda^{p,q}(X) \to \Lambda^{p+1,q}(X) \text{ and } \overline{\partial}\colon \Lambda^{p,q}(X) \to \Lambda^{p,q+1}(X).$$

Let $V \to X$ be a holomorphic vector bundle and

$$\Lambda^{p,q}(X,V) = \Gamma(\Lambda^p(T^{1,0}X)^* \otimes \Lambda^q(T^{0,1}X)^* \otimes V).$$

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Let $\Omega \subset X$ be a trivialization chart of $V \to X$, i.e. there is a biholomorphic map ψ such that $\psi\colon V{\restriction_\Omega} \xrightarrow{\cong} \Omega \times \mathbb{C}^k$. Let $e_1, \dots, e_k$ be a local holomorphic frame: $\{e_i \mid 1 \le i \le k\} \in \Gamma_{\mathrm{hol}}(V{\restriction_\Omega})$ such that $e_1(x), \dots, e_k(x) \in V_x$ is a basis for all $x \in \Omega$. Then $\Lambda^{p,q}(\Omega, V{\restriction_\Omega}) \cong \Lambda^{p,q}(\Omega, \mathbb{C}^k)$ and $\omega \in \Lambda^{p,q}(\Omega, V{\restriction_\Omega})$ have the following local form

$$\omega = \sum_{i=1}^{k} \omega_i \otimes e_i$$

Let $\bigcup_j \Omega_j$ be a good covering of X and $\{\chi_j\}$ the assoziated partion of unity. We define $\omega \in \Lambda^{p,q}(X, V)$ by gluing the local (p,q)-forms $\omega^j = \omega{\restriction_{\Omega_j}} \in \Lambda^{p,q}(\Omega_j, V{\restriction_{\Omega_j}})$ via χ_j:

$$\omega = \sum_j \chi_j \omega^j = \sum_j \chi_j \Big(\sum_{i=1}^{k} \omega_i^j \otimes e_i \Big).$$

By assumption is the transformation map ϕ of local frames $e_1, \dots, e_k$ and $e'_1, \dots, e'_k$ holomorphic, so we define an elliptic complex

$$0 \to \Lambda^{p,0}(X, V) \xrightarrow{\overline{\partial}} \Lambda^{p,1}(X, V) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Lambda^{p,n}(X, V) \to 0, \tag{3}$$

where $\overline{\partial}\omega = \sum_i (\overline{\partial}\omega_i) \otimes e_i$.

Let $\mathcal{O}(V)$ be the sheaf of germs of holomorphic sections of V. On the sheaf level there is a fine resolution of $\mathcal{O}(V)$:

$$0 \to \mathcal{O}(V) \to \mathcal{A}^{0,0}(V) \to \mathcal{A}^{0,1}(V) \to \cdots \to \mathcal{A}^{0,n}(V) \to 0,$$

where $\mathcal{A}^{0,q}(V)$ is sheaf of germs of sections of $\Lambda^{0,q} \otimes V$, such that $H^{0,q}(X; V) \cong H^q(X; \mathcal{O}(V))$ and by (2) $H^{p,q}(X; V) \cong H^q(X; \mathcal{O}(\Lambda^{p,0} \otimes V))$.

We consider now a holomorphic map $f\colon X \to X$. The natural lifting of f to $\Lambda^*(X)$ is then compatible with $\overline{\partial}$ and therefore induces endomorphisms $\Lambda^{p,*}f$ in each complex $\Lambda^{p,*}(X)$. To lift f to the complex $\Lambda^*(X, V)$, one only needs a *holomorphic* bundle homomorphism $\varphi\colon f^*V \to V$. In terms of it

$$\Lambda^{0,q} f \otimes \varphi\colon f^*(\Lambda^{0,q} \otimes V) \to \Lambda^{0,q} \otimes V \qquad (0 \le q \le n).$$

The coresponding endomorphism in the sheaf cohomology $H^q(X; \mathcal{O}(V)) \cong H^{0,q}(X; V)$ will be denoted by $(f \otimes \varphi)_!$ so that the Lefschetz numbers of $\Lambda^{0,q} f \otimes \varphi$ are given by:

$$L(\Lambda^{0,*} f \otimes \varphi) = \sum_{q=0}^{n} (-1)^q \operatorname{Tr}((f \otimes \varphi)_! {\restriction} H^{0,q}(X; V)).$$

Theorem 1. *Let X be a compact complex manifold and let $V \to X$ a holomorphic vector bundle. Further let $f\colon X \to X$ be a holomorphic map with simple fixed points and $\varphi\colon f^*V \to V$ a holomorphic bundle homomorphism. Then the Lefschetz number $L(\Lambda^{0,*} f \otimes \varphi)$ of $H^*(X; \mathcal{O}(V))$ is:*

$$L(\Lambda^{0,*} f \otimes \varphi) = \sum_{z \in \mathrm{Fix}(f)} \frac{\mathrm{Tr}_{\mathbb{C}}\, \varphi_z}{\det_{\mathbb{C}}(\mathbb{1} - \partial f_z)}. \tag{4}$$

2. Geometric methods in representation theory

A Lie algebra $\mathfrak{g}$ is semisimple if it can be written as a direct sum of simple ideals.

Remark. One can consider a linear reductive Lie algebra $\mathfrak{g}$, which generalizes the consideration of semisimple Lie algebras, since $\mathfrak{g}$ may be written as a direct sum of ideals

$$\mathfrak{g} = Z_{\mathfrak{g}} \oplus [\mathfrak{g}, \mathfrak{g}],$$

with $Z_{\mathfrak{g}}$ is the centre of $\mathfrak{g}$ and $[\mathfrak{g}, \mathfrak{g}]$ is semisimple Lie algebra. For the reason of simplicity i will consider only semisimple Lie algebra.

Maximal compact subgroups and Cartan decomposition. Let G be a connected semisimple Lie group. We denote by $K \subset G$ a *maximal compact subgroup.* The maxiamal compact subgroups of G have the following properties:

1) any two maximal compact subgroups of G are conjugate by an element of G;
2) the normalizer of K in G coincides with K, i.e., $N_G(K) = K$.

Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K respectively and K acts on $\mathfrak{g}$ via the restriction of the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$, $\mathrm{Ad}(g)(Y) = g^{-1}Yg$.

Let $\theta\colon \mathfrak{g} \to \mathfrak{g}$ be a *Cartan involution* of $\mathfrak{g}$, i.e., there exists a unique K-invariant linear complement $\mathfrak{p} = \mathcal{E}(\theta; -1)$ of $\mathfrak{k} = \mathcal{E}(\theta; 1)$ in $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \tag{5}$$

with the following property $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$.

Example. The group $G = \mathrm{SL}(n, \mathbb{R})$ contains $K = \mathrm{SO}(n)$ a maximal compact subgroup. In this situation

$$\begin{aligned}
\mathfrak{g} &= \{Y \in \mathrm{End}(\mathbb{R}^n) \mid \mathrm{tr}(Y) = 0\},\\
\mathfrak{k} &= \{Y \in \mathrm{End}(\mathbb{R}^n) \mid Y^\top + Y = 0, \quad \mathrm{tr}(Y) = 0\},\\
\mathfrak{p} &= \{Y \in \mathrm{End}(\mathbb{R}^n) \mid Y^\top - Y = 0, \quad \mathrm{tr}(Y) = 0\}.
\end{aligned}$$

On the Lie algebra level a Cartan involution is $\theta(Y) = -Y^\top$ and on the group level $\theta(g) = (g^\top)^{-1}$. The group K can be described as the fix point set of θ, i.e., $K = \{g \in G \mid \theta(g) = g\}$.

Complexifications of linear groups. Let G be a connected linear Lie group and let $\mathfrak{g} = \mathrm{Lie}(G)$ be its Lie algebra. Like any linear Lie Group, G has a *complexification* – a complex Lie group $G^{\mathbb{C}}$, with Lie algebra $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C}$ containing $G \hookrightarrow G^{\mathbb{C}}$ as a Lie subgroup, such that $\mathfrak{g} \hookrightarrow \mathfrak{g}_{\mathbb{C}}$, $Y \mapsto Y \otimes 1$. When $G^{\mathbb{C}}$ is a complexification of G, one calls G a *real form* of $G^{\mathbb{C}}$. One can complexify the Cartan decomposition (5): $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$, where $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \otimes \mathbb{C}$ and $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p} \otimes \mathbb{C}$. The complexification $G^{\mathbb{C}}$ of G contains naturally $K^{\mathbb{C}} = \mathrm{Exp}(\mathfrak{k})$ as complex Lie subgroup.

Remark. A complexification $K^{\mathbb{C}}$ of K can not be compact unless $K = \{e\}$, which does not happen unless G is abelian. Indeed, any non-zero $Y \in \mathfrak{k}$ is diagonalizable over $\mathbb{C}$, with pure imaginary eigenvalues. So the complex one-parameter subgroup $\{z \mapsto \exp(zY)\}$ of $K^{\mathbb{C}}$ is unbounded.

By construction, the Lie algebras $\mathfrak{g}$, $\mathfrak{k}$ its complexifications and the corresponding Lie groups satisfy the following containments:

$$\begin{array}{ccc} \mathfrak{g} & \subset & \mathfrak{g}_{\mathbb{C}} \\ \cup & & \cup \\ \mathfrak{k} & \subset & \mathfrak{k}_{\mathbb{C}} \end{array} \quad \text{and} \quad \begin{array}{ccc} G & \subset & G^{\mathbb{C}} \\ \cup & & \cup \\ K & \subset & K^{\mathbb{C}}. \end{array}$$

Since $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$ and $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$,

$$\mathfrak{u} := \mathfrak{k} \oplus \mathbb{i}\mathfrak{p}$$

is a real Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Let U denote Lie subgroup of $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{u}$. Since G is a semisimple Lie group by assumption we know that U is compact. Thus U lies in a maximal compact subgroup of $G^{\mathbb{C}}$, which we denote also by U. Since $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{u} \oplus \mathbb{i}\mathfrak{u}$ a maximal compact subgroup U is a real form of $G^{\mathbb{C}}$ and $K = U \cap G^{\mathbb{C}}$. Thus we call U also a *compact real form* of $G^{\mathbb{C}}$.

Example. Let $G = \mathrm{SL}(n,\mathbb{R})$, $K = \mathrm{SO}(n)$. The complexifications are: $G^{\mathbb{C}} = \mathrm{SL}(n,\mathbb{C})$ and $K^{\mathbb{C}} = \mathrm{SO}(n,\mathbb{C})$. The corresponding compact real form of $G^{\mathbb{C}}$ is then $U = \mathrm{SU}(n)$.

Since $\mathfrak{g}_{\mathbb{C}} = \mathfrak{u} \otimes \mathbb{C}$, these two Lie algebras have the same representations over $\mathbb{C}$. On the global level this means

$$\left\{ \begin{array}{c} \text{holomorphic finite dimensional} \\ \text{representations of } G^{\mathbb{C}} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{finite dimensional complex} \\ \text{representations of } U \end{array} \right\}; \tag{6}$$

this bijection one calls *Weyl unitary trick.* Since on every compact group U there is a left invariant Haar measure $\mathrm{d}u$, any representation of U can be made unitary. This implies that:

finite dimensional representations of a compact group are completely reducible.

In particular, to understand the finite dimensional representations of U, it suffices to understand the finite dimensional, irreducible representations of U over $\mathbb{C}$ up to a isomorphism, i.e., $\mathrm{Irr}_{\mathbb{C}}(U)$.

Complex semisimple Lie algebras. Let $\mathfrak{g}_{\mathbb{C}}$ be a complex Lie algebra, then by Cartan criterior for semisimplicity $\mathfrak{g}_{\mathbb{C}}$ is semisimple iff the Killing form $B(Y,Y') := \mathrm{Tr}(\mathrm{ad}(Y)\,\mathrm{ad}(Y'))$ on $\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}$ is nondenegenerate. A *Cartan subalgebra* $\mathfrak{h}_{\mathbb{C}}$ is in this case a maximal abelian subspace of $\mathfrak{g}_{\mathbb{C}}$ in which every $\mathrm{ad}(Z)$ for $Z \in \mathfrak{h}_{\mathbb{C}}$ is diagonable.
The elements $\alpha \in \mathfrak{h}_{\mathbb{C}}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}_{\mathbb{C}}, \mathbb{C})$ are *roots* and $\mathfrak{g}^{\alpha}$ are *root spaces*, the α being defined as the nonzero elements of $\mathfrak{h}_{\mathbb{C}}^*$ such that

$$\mathfrak{g}_{\mathbb{C}}^{\alpha} = \{Y \in \mathfrak{g}_{\mathbb{C}} \mid \mathrm{ad}(Z)(Y) = [Z,Y] = \alpha(Z)Y \text{ for all } Z \in \mathfrak{h}_{\mathbb{C}}\}$$

is nonzero. Let Φ be the set of all roots.

Example. Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(n,\mathbb{C}) = \{Y \in \mathrm{Mat}_n(\mathbb{C}) \mid \mathrm{tr}(Y) = 0\}$. The Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ is the space of diagonal matices in $\mathfrak{g}_{\mathbb{C}}$.

For a complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ there is a decompositions of the form

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_{\mathbb{C}}^{\alpha} \tag{7}$$

and have the following properties:

1) $[\mathfrak{g}_{\mathbb{C}}^{\alpha}, \mathfrak{g}_{\mathbb{C}}^{\beta}] \subseteq \mathfrak{g}_{\mathbb{C}}^{\alpha+\beta}$ and $[\mathfrak{g}_{\mathbb{C}}^{\alpha}, \mathfrak{g}_{\mathbb{C}}^{\beta}] = \mathfrak{g}_{\mathbb{C}}^{\alpha+\beta}$ if $\alpha+\beta \neq 0$;
2) $B(\mathfrak{g}_{\mathbb{C}}^{\alpha}, \mathfrak{g}_{\mathbb{C}}^{\beta}) = 0$ for $\alpha, \beta \in \Phi \cup \{0\}$ and $\alpha + \beta \neq 0$;
3) $B\restriction_{\mathfrak{h}_{\mathbb{C}}\times\mathfrak{h}_{\mathbb{C}}}$ is nondegenerate. Define Z_α to be the element of $\mathfrak{h}_{\mathbb{C}}$ paired with α;
4) If α is in Φ, then $\dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}}^{\alpha} = 1$;
5) The real subspace $\mathfrak{h}$ of $\mathfrak{h}_{\mathbb{C}}$ on which all roots are real is a real form of $\mathfrak{h}_{\mathbb{C}}$, and $B\restriction_{\mathfrak{h}\times\mathfrak{h}}$ is an inner product.

The centralizer $H = Z_{G^{\mathbb{C}}}(\mathfrak{h}_{\mathbb{C}})$ is a Cartan subgroup of $G^{\mathbb{C}}$. It is connected since $G^{\mathbb{C}}$ is complex, define

$$\widehat{H} \overset{\text{def}}{=} \mathrm{Hom}_{\mathbb{C}}(H, S^1)$$

the group of holomorphic homomorphisms from H to the multiplicative group $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. It is an abelian group, which we identify with the *weight lattice* $\Lambda \subset \mathfrak{h}_{\mathbb{C}}^*$, i.e., the lattice of linear functionals on $\mathfrak{h}_{\mathbb{C}}^*$ whose values on the *unit lattice*

$$L = \{Z \in \mathfrak{h}_{\mathbb{C}} \mid \exp(Z) = e\}$$

are integral multiples of $2\pi\mathbb{i}$. Explicitly, the identification $\Lambda \cong \widehat{H}$ is given by

$$\lambda \overset{1:1}{\longleftrightarrow} e^{\lambda},$$

with $e^{\lambda}(\exp(Z)) = e^{\langle\lambda, Z\rangle}$ for $Z \in \mathfrak{h}_{\mathbb{C}}$; here $\langle\lambda, Z\rangle$ refers to the canonical pairing between $\mathfrak{h}_{\mathbb{C}}^*$ and $\mathfrak{h}_{\mathbb{C}}$ induced by the Killing form restricted to a Cartan subalgebra.

Maximal Tori and the weight lattice. Let U be a *connected* compact semisimple Lie group defined as above and $T \subset U$ be a *maximal torus*. Since any two maximal tori in U are conjugated by an element of U, we fix a maximal torus T of U and denote by $\mathfrak{t}$ its Lie algebra. Since T is abelian and connected, the exponential map $\exp\colon \mathfrak{t} \to T$ is a surjective homomorphism, moreover this map is locally bijective, hence a covering homomorphism

$$\exp\colon \mathfrak{t}/L_T \xrightarrow{\cong} T,$$

where $L_T = \{Z \in \mathfrak{t} \mid \exp Z = e\} \subset \mathfrak{t}$ a discrete cocompact subgroup, i.e., the unit lattice. Let $\widehat{T}$ denote the group of characters, i.e., the group of homomorphisms from T to the unit circle S^1. Then the weight lattice $\Lambda \subset \mathbb{i}\mathfrak{t}$

$$\Lambda := \{\lambda \in \mathbb{i}\mathfrak{t}^* \mid \langle\lambda, L_T\rangle \subset 2\pi\mathbb{i}\mathbb{Z}\} \xrightarrow{\cong} \widehat{T}, \quad \lambda \mapsto e^{\lambda},$$

with $e^{\lambda}\colon T \to S^1$ defined by $e^{\lambda}(\exp(Z)) = \mathrm{e}^{\langle\lambda, Z\rangle}$ for any $Z \in L_T$ is the dual lattice of the unit lattice $L_T \subset \mathfrak{t}$.

The space of roots $\Phi = \Phi(U)$ of U are by definition the characters of the irredicible representation into which the tangent space of U/T at the coset $eT \in U/T$ decomposes under the left action of T, i.e., in Lie algebra terms we have with the identification $\Lambda \cong \widehat{T}$

$$(\mathfrak{u}/\mathfrak{t}) \otimes \mathbb{C} \cong \sum_{\alpha} E_\alpha \overset{(7)}{=} \sum_{\alpha\in\Phi} \mathfrak{g}_{\mathbb{C}}^{\alpha}. \tag{8}$$

Since $\Phi \subset \Lambda - \{0\} \subset \mathbb{i}\mathfrak{t}^* \subset \mathfrak{h}_{\mathbb{C}}^*$, roots take pure imaginary values on the real Lie algebra $\mathfrak{t}$, which implies $\overline{\mathfrak{g}_{\mathbb{C}}^{\alpha}} = \mathfrak{g}_{\mathbb{C}}^{\overline{\alpha}} = \mathfrak{g}_{\mathbb{C}}^{-\alpha}$. For this reason every root α occurs with the inverse $-\alpha$, so that it is natural to partion Φ into a positive set of roots Φ^+ und their inverse into negative set of roots $\Phi = \Phi^+ \sqcup \Phi^-$. Of course this choise is to be made with some compatibility relative to the Lie structure of $\mathfrak{u} \otimes \mathbb{C} = \mathfrak{g}_\mathbb{C}$; that is, one would like the relation

$$[E_\alpha, E_\beta] \subseteq E_{\alpha+\beta} \tag{9}$$

to hold whenever α, β and $\alpha + \beta$ are in Φ^+. Weyl shows that such choise of Φ^+ do exists and in fact that they are in $1 : 1$ correspondence with the *dominant Weyl chambers* into which the action of the Weyl group

$$W = N_U(T)/Z_U(T) = N_U(T)/T$$

breaks up $\mathfrak{t}$.

The compatibility condition (9) one can interpret in its more geomerical form, namely as an integrability condition for a homogeneous complex structure on U/T. Indeed a choise of Φ^+ induces an almost complex structure on U/T by declaring that the E_α, $\alpha > 0$, generate the holomorphic part of the tangent space of U/T at $o := eT \in U/T$, i.e. $T_o^{1,0}(U/T)$. By the group action one translate this subspace to the holomorphic part of the tangent space of U/T at $\mathfrak{x} \in U/T$.

A fundamental fact in the theory of compact groups is the following extension of the spectral theorem:

Every $u \in U$ is conjugated to an element of T.

It follows that functions f on U are determinated by their values $\iota^* f$ on T alone (where $\iota\colon T \hookrightarrow U$) and it therefore stands to reason that if $\mathrm{d}u$ denotes the left invariant Haar measure on U, then there must be a measure $\mathrm{d}\mu$ on T with the prorerty

$$\int_U f \mathrm{d}u = \int_T \iota^* f \mathrm{d}\mu$$

for all integrable functions f on U. H. Weyl now finds an expilicit formula for $\mathrm{d}\mu$ in terms of the positive roots and the Weyl group

$$\mathrm{d}\mu = \frac{1}{\#W} |D|^2 \mathrm{d}t,$$

with $D = \prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2})$. Furthermore this D is not only well defined, but is antisymmetric as regards the action of W on Λ, and so can also be described in the followig way:

$$D = \sum_{w \in W} \mathrm{sign}(w) e^{w(\rho)},$$

where $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ and $\mathrm{sign}(w) = D^w / D \in \{\pm 1\}$.

Remark. To compute the Weyl denominator D in this way one needs the assumption $G^\mathbb{C}$ to be *simply connected*. This condition is of course equvalent to the assumption U to be simply connected, since $U \hookrightarrow G^\mathbb{C}$ is a deformations retract by global Cartan decompostion, so $\pi_1(G^\mathbb{C}) = \pi_1(U)$. Then only in this case $\rho = \sum_{\alpha \in \Phi^+} \alpha$ lies in Λ, such that the product of positive roots $\prod_{\alpha \in \Phi^+} e^\alpha$ have a square root, which is given by $e^\rho = e^{\frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha}$.

At this moment one can see the deeper reason why the character of a finite dimensional complex irreducible representation can be compute by restriction on a maximal torus T of U. Consider the charcter of a finite dimensional complex irreducible representation as an element of $C^0(U) = \{f\colon U \to \mathbb{C} \mid f \text{ continous}\}$ defined by

$$U \ni x \mapsto \operatorname{Tr}(\pi(x)).$$

Now since $\operatorname{Tr}(\pi(x)) = \operatorname{Tr}(\pi(gxg^{-1}))$ for any $g \in U$, and since every $u \in U$ is conjugated to an element of T we conclude, that $\operatorname{Tr}\pi = \operatorname{Tr}\pi\restriction_T$

Highst weight theorem and $\operatorname{Irr}_{\mathbb{C}}(\mathfrak{g})$. An element $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ is said to be *singular*, if $\langle\alpha,\lambda\rangle = 0$ for some $\alpha \in \Phi$, ond otherwise *regular*. The set of regular elements in $\mathbb{i}\mathfrak{t}^*$ breacks up into a finite, disjoint union of open, convex cones, the so-called *Weyl chambers*. The Weyl chamber C can be recovered from the system of positive roots Φ^+, which we call dominant Weyl chamber,

$$C = \{Z \in \mathbb{i}\mathfrak{t} \mid \langle\alpha, Z\rangle > 0\} \xleftrightarrow{1:1} \Phi^+.$$

Definition. An element $\lambda \in \mathbb{i}\mathfrak{t}^*$ is said to be *dominant* if $\langle\lambda,\alpha\rangle \geq 0$ for all $\alpha \in \Phi^+$.

Via the identification $\mathbb{i}\mathfrak{t}^* \cong \mathbb{i}\mathfrak{t}$ by the Killing form, the set of all dominant regular $\lambda \in \mathbb{i}\mathfrak{t}^*$ corresponds precisely to the dominant Weyl chamber C. Since the Weyl group W acts simply transitively on the set of Weyl chambers, every regular $\lambda \in \mathbb{i}\mathfrak{t}^*$ is W-conjugated to exactly one dominant regular $\lambda' \in \mathbb{i}\mathfrak{t}^*$. The action of W preseves the weight lattice Λ, hence every $\lambda \in \Lambda$ is W-conjugate to a unique dominant $\lambda' \in \Lambda$, in other words

$$\{\lambda \in \Lambda \mid \lambda \text{ is dominant}\} \cong W\backslash\Lambda.$$

Now by the theorem of the hights weight, which says that for every $\pi \in \operatorname{Irr}_{\mathbb{C}}(U)$ there is exactly one weight λ, such that $\lambda + \alpha$ is not a weight for any $\alpha \in \Phi^+$, the highest weight of π. The heightst weight is dominant, has the multiplicity one, i.e., $\dim_{\mathbb{C}} V_\lambda = 1$, and determinates the representation π up to an isomorphism. Every dominant $\lambda \in \Lambda$ aries as the highst weight of an irreducible representation π.

In effect, the theorem parametrize the isomorphism classes of irreducible finite dimensional representation over $\mathbb{C}$ in terms of their heighst weights:

$$\operatorname{Irr}_{\mathbb{C}}(U) \xleftrightarrow{1:1} \{\lambda \in \Lambda \mid \lambda \text{ is dominant}\} \xleftrightarrow{1:1} W\backslash\Lambda.$$

3. Geometric Realization of $\operatorname{Irr}_{\mathbb{C}}(U)$

A Borel subalgebra $\mathfrak{b}$ is a maximal solvable subalgebra of $\mathfrak{g}_{\mathbb{C}}$ of the form $\mathfrak{b} = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is $\sum_{\alpha\in\Phi^+} \mathfrak{g}^{-\alpha}$, Φ^+ is a system of positive roots of $\mathfrak{h}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$. Any two Borel subalgebras are $\operatorname{Ad}(G)$-conjugated. To define the notion of Borel subgroups, let us consider a particular Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}_{\mathbb{C}}$. Its normalizer in G,

$$B = N_{G^{\mathbb{C}}}(\mathfrak{b}) = \{g \in G^{\mathbb{C}} \mid \operatorname{Ad}(g)\mathfrak{b} \subseteq \mathfrak{b}\}$$

is connected and has Lie algebra $\mathfrak{b}$. Groups of this type are called *Borel subgroups* of $G^{\mathbb{C}}$. It should be remarked that the connectedness of Borel subgroups depends crucially on the assumption that the ambient group $G^{\mathbb{C}}$ is complex. As a set, the flag variety X of $\mathfrak{g}_{\mathbb{C}}$ is the collection of all Borel subalgebras of $\mathfrak{g}_{\mathbb{C}}$. The solvable subalgebras of a given dimension constitute a closed subvariety in a Grassmannian for $\mathfrak{g}_{\mathbb{C}}$, hence X has a natural structure of complex projective variety. Since any two Borel subalgebras are conjugate via Ad, $G^{\mathbb{C}}$ acts transitively on X, with isotropy group $B = N_{G^{\mathbb{C}}}(\mathfrak{b})$ at the point at $\mathfrak{b}$. Consequently we may make the identification $X \cong G^{\mathbb{C}}/B$. Every complex algebraic variety is smooth (i.e., nonsingular) outside a proper subvariety. But $G^{\mathbb{C}}$ acts transitively on X, so the flag variety cannot have any singularities: it is a smooth complex projective variety.

Example. Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(n,\mathbb{C})$. Then X is (naturally isomorphic to) the variety of all complete flags in $\mathbb{C}^n$, i.e., nested sequences of linear subspaces of $\mathbb{C}^n$, one in each complex dimension, i.e., $\dim_{\mathbb{C}}(F_j/F_{j-1}) = 1$:

$$X \cong \{(F_j) \mid 0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^n \text{ and } \dim F_j = j\}.$$

To see this, we assign to the complete flag (F_j) its stabilizer in $\mathfrak{sl}(n,\mathbb{C})$, which turns out to be a Borel subalgebra $\mathfrak{b}$; this can be checked by looking at any particular flag (F_j), since any two are conjugate under the action of $G^{\mathbb{C}} = \mathrm{SL}(n,\mathbb{C})$. Using the transitivity of the $G^{\mathbb{C}}$-action on the set of complete flags once more, we get the identification between this set and $G^{\mathbb{C}}/N_{G^{\mathbb{C}}}(\mathfrak{b}) \cong G^{\mathbb{C}}/B \cong X$.

Each member e^λ of $\widehat{H}$ lifts to a holomorphic character $e^\lambda\colon B \to \mathbb{C}^\times$ via the isomorphism $H \cong B^{\mathrm{ab}} = B/[B,B]$. Consider the fiber bundle product

$$L_\lambda = G^{\mathbb{C}} \times_B \mathbb{C}_\lambda,$$

where $\mathbb{C}_\lambda$ denotes $\mathbb{C}$, equipped with the B-action via the character e^λ. By definition, the fiber product L_λ is the quotient $G^{\mathbb{C}} \times \mathbb{C}_\lambda/\sim$ under the equvalence relation

$$(gb, z) \sim (g, e^\lambda(b)z).$$

The natural projection $G^{\mathbb{C}} \times_B \mathbb{C}_\lambda \to G^{\mathbb{C}}$ induces a well defined $G^{\mathbb{C}}$-equivariant holomorphic map $L_\lambda \to G^{\mathbb{C}}/B \cong X$, which exihibits L_λ as a $G^{\mathbb{C}}$-equivariant holomorphic line bundle over X, i.e., a holomorphic line bundle with a holomorphic $G^{\mathbb{C}}$-action (by bundle maps) that lies over the action of $G^{\mathbb{C}}$ on the base space X. Let us summerize the previuos results

$$\widehat{T} \cong \left\{\begin{matrix}\text{holomorphic}\\ \text{characters on } H\end{matrix}\right\} \cong \left\{\begin{matrix}\text{holomorphic}\\ \text{characters on } B\end{matrix}\right\} \cong \left\{\begin{matrix}\text{holomorphic } G^{\mathbb{C}}\text{-equivariant}\\ \text{line bundles over } X \cong G^{\mathbb{C}}/B\end{matrix}\right\}.$$

Identifying the dual group $\widehat{T}$ with the weight lattice Λ as usual, we get a canonical isomorphism

$$\Lambda \cong \left\{\begin{matrix}\text{group of holomorphic } G^{\mathbb{C}}\text{-equivariant}\\ \text{line bundles over } X \cong G^{\mathbb{C}}/B\end{matrix}\right\}, \quad \lambda \overset{1:1}{\longleftrightarrow} L_\lambda.$$

The action of $G^{\mathbb{C}}$ on X and L_λ determines a holomorphic, linear action on the space of global section $H^0(X;\mathcal{O}(L_\lambda))$ and, by functorality, also on the higher cohomology groups $H^q(X;\mathcal{O}(L_\lambda)) \cong H^{0,q}(X;L_\Lambda)$, $q > 0$. These groups are finite dimensional since X is compact. The Borel-Weil theorem describes the resulting representations of the compact real form $U \subset G^{\mathbb{C}}$, and in view of (6), also as holomorphic representation of $G^{\mathbb{C}}$.

Theorem 2 (Borel-Weil)**.** *If λ is a dominant weight, the representation of U on $H^0(X;\mathcal{O}(L_\lambda))$ is irreducible, of highst weight λ, and $H^q(X;\mathcal{O}(L_\lambda)) = 0$ for $q > 0$. If λ fails to be dominant, then $H^0(X;\mathcal{O}(L_\lambda)) = 0$.*

3.1. **Sketch of the Proof of Borel-Weil theorem.** Let $U \hookrightarrow G^{\mathbb{C}}$ be a compact real form, i.e., a compact Lie subgroup with Lie algebra $\mathfrak{u}$ such that $\mathfrak{g} = \mathfrak{u} \oplus i\mathfrak{u}$. We can choose the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_{\mathbb{C}}$ so that it is the complexification of a subalgebra $\mathfrak{t}$ of $\mathfrak{u}$; all we have to do is take $\mathfrak{t}$ to be any maximal abelian subspace of $\mathfrak{u}$. Then $T = U \cap H$ is a Cartan subgroup of U, i.e., a maximal torus.

The U-orbit of the point $\mathfrak{b}$ of X is a closed submanifold because U is compact, and it is open in X by a dimension count. Therefore U acts transitively on X. To compute the isotropy subgroup at $\mathfrak{b}$, we observe that $U \cap B = U \cap B \cap \overline{B} = U \cap H = T$, hence

$$X \cong G^{\mathbb{C}}/B \cong U/(U \cap B) = U/T.$$

If we identify $X \cong U/T$, we see that L_λ, as U-equivariant complex C^∞-line bundle, is given by

$$L_\lambda \cong U \times_T \mathbb{C}_\lambda, \tag{10}$$

here $\mathbb{C}_\lambda$ is the one dimensional T-module on which T acts via the character e^λ. This leads to the following description of the space of C^∞-sections of L_λ:

$$C^\infty(X, L_\lambda) \cong \{f \in C^\infty(U) \mid f(gt) = e^{-\lambda}(t)f(g) \text{ for all } t \in T\} \cong (C^\infty(U) \otimes \mathbb{C}_\lambda)^T, \tag{11}$$

here $(C^\infty(U) \otimes \mathbb{C}_\lambda)^T$ denotes the space of T-invariants in $C^\infty(U) \otimes \mathbb{C}_\lambda$, relative to the action by right translation on $C^\infty(U)$ and by e^λ on $\mathbb{C}_\lambda$. How can one characterize the holomorphic sections among the C^∞-sections – in other words, what are the Cauchy-Riemann equations? Suppose that $\Omega \subset X \cong U/T$ is open and that $\widetilde{\Omega} \subset U$ is its inverse image. Then

$$C^\infty(\Omega, L_\lambda) \cong \{f \in C^\infty(\widetilde{\Omega}) \mid f(gt) = e^{-\lambda}(t)f(g) \text{ for } t \in T\} \tag{12}$$

by specialization of the previous isomorphism to Ω, and our question is answered by:

Lemma 3. *Under the isomorphism* (12)*, a function f on $\widetilde{\Omega}$ corresponds to a holomorphic section of L_λ over Ω if and only if $R(\xi)f = 0$ for all $\xi \in \mathfrak{n}$, where $R(\xi)$ denotes infinitesimal right translation on U by $\xi \in \mathfrak{g}_{\mathbb{C}} = \mathfrak{u} \oplus i\mathfrak{u}$.*

The lemma is readily proved by starting from the Cauchy-Riemann equations on $G^{\mathbb{C}}$. Using it, we can identify the space of global holomorphic sections as

$$H^0(X; \mathcal{O}(L_\lambda)) \cong \{f \in C^\infty(U) \mid R(\mathfrak{n})f = 0 \text{ and } f(gt) = e^{-\lambda}(t)f(g) \text{ for } t \in T\}$$

and this isomorphism is an isomorphism of representations of U. The space $C^\infty(U)$ is contained in $L^2(U)$, which we can identify by the Peter-Weyl theorem as a Hilbert space direct sum $\sum_{i \in \widehat{U}} V_i \widehat{\otimes} V_i^*$. Here U acts on V_i by left translation, and on V_i^* by right translation. The subspace of $C^\infty(U)$ corresponding to $H^0(X; \mathcal{O}(L_\lambda))$ is finite dimensional and U-invariant, hence contained in the *algebraic* direct sum $\bigoplus_{i \in \widehat{U}} V_i \otimes V_i^*$. We conclude that

$$\begin{aligned} H^0(X; \mathcal{O}(L_\lambda)) &\cong \Big\{f \in \textstyle\bigoplus_i V_i \otimes V_i^* \mid R(\mathfrak{n})f = 0 \text{ and } f(gt) = e^{-\lambda}(t)f(g) \text{ for } t \in T\Big\} \\ &\cong \bigoplus_i V_i \otimes \big\{v \in (V_i^* \otimes \mathbb{C}_\lambda)^T \mid \mathfrak{n}\, v = 0\big\} \end{aligned}$$

The condition $\mathfrak{n}v = 0$ picks out the *lowest* weight space since $\mathfrak{b}$ is built from the root spaces for the negative roots. Therefore the right side is

$$\bigoplus_{\substack{V_i^* \text{ has lowest} \\ \text{weight } -\lambda}} V_i \otimes (\text{lowest weight space in } V_i^*).$$

At this point, the description of $H^0(X;\mathcal{O}(L_\lambda))$ in Borel-Weil theorem can be deduced from the theorem of the highest weight and the vanishing of the higher cohomology groups is a consequence of the Kodaira vanishing theorem.

Remark. According to our convention, $\mathfrak{b}$ is built from the root spaces for the negative roots. This has the effect of making the line bundle L_λ "positive" in the sense of complex analysis (see [Wel80, p. 223], for example) precisely when the parameter λ is dominant. The opposite convention, which uses the root spaces for positive roots, lets positive line bundles correspond to antidominant weights and makes $H^0(X;\mathcal{O}(L_\lambda))$, for antidominant λ, the $G^{\mathbb{C}}$-module with lowest weight λ.

We denote by ρ_λ the by e^λ induced irreducible highest weight representation of U:

$$\rho_\lambda = \operatorname{Ind}_T^U(e^\lambda)\colon U \to \operatorname{GL}(H^0(X;\mathcal{O}(L_\lambda))), \quad (\rho_\lambda(u)f)(x) = f(u^{-1}x), \tag{13}$$

on the space of global holomorphic sections of L_λ, i.e., on $H^0(X;\mathcal{O}(L_\lambda))$.

4. Proof of the Weyl character formula

Let $\pi\colon U \to U/T$ be the canonical projection. We start with the function $f\colon X \to X$ which have to be the left translation of each element of $\mathfrak{x} \in X$ by $g^{-1} \in U$ defined by

$$l_{g^{-1}}\colon X \to X, \quad l_{g^{-1}}(\mathfrak{x}) = g^{-1}.\mathfrak{x} = g^{-1}.\pi(x) = \pi(g^{-1}x),$$

where $\mathfrak{x} = \pi(x)$ denotes a coset in U/T. Now let $\lambda \in \Lambda$ be a highest weight, $\mathbb{C}_\lambda$ is the one dimensional T-module on which T acts via the character e^λ. Let L_λ be the associated homogeneous line bundle:

$$L_\lambda = U \times_T \mathbb{C}_\lambda \to U/T,$$

where $U \times_T \mathbb{C}_\lambda = (U \times \mathbb{C}_\lambda)/\sim$ and the equvalence relation $\sim$ is given by $(ut, z) \sim (u, \mathrm{e}^\lambda(t)z)$. Let $L_g\colon U \to U$ be the left translation on U by $g \in U$. Clearly $L_g \times \mathbb{1}\colon U \times \mathbb{C}_\lambda \to U \times \mathbb{C}_\lambda$ preserves the fibers of $U \times \mathbb{C}_\lambda \to L_\lambda$ and hence induces a map

$$\varphi_g := L_g \times_T \mathbb{1}\colon L_\lambda \to L_\lambda,$$

which maps the fiber over $l_{g^{-1}}(\mathfrak{x}) = \pi(g^{-1}x)$ lineary into the fiber over $\mathfrak{x} = \pi(x)$, i.e.,

$$\varphi_g\colon (L_\lambda)_{\pi(g^{-1}x)} \to (L_\lambda)_{\pi(x)}.$$

One may interpreted φ_g as a lifting of the map $l_{g^{-1}}$ on U/T to the associated homogeneuos line bundle L_λ over U/T, i.e., for $[g^{-1}x, z] \in (L_\lambda)_{\pi(g^{-1}x)}$:

$$\varphi_g([g^{-1}u, z]) = (L_g \times_T \mathbb{1})([g^{-1}x, z]) = [L_g(g^{-1}x), z] = [x, z] \in (L_\lambda)_{\pi(x)}.$$

Consider now a *fixed point* $\mathfrak{x} \in X$ of $l_{g^{-1}}$, i.e., by definition that for each point x in the coset $\mathfrak{x} = \pi(x)$ we must have the relation

$$g^{-1}x = xh_g(x) \tag{14}$$

for some $h_g(x) \in T$. Conversely if (14) holds for some $t \in T$, then $\pi(x) = \mathfrak{x}$ is a fixed point of $l_{g^{-1}}\colon U/T \to U/T$. Hence we get the following

Lemma 4. *$l_{g^{-1}}$ has a fixed point iff g contained in the orbit of T under the conjugation action of G, i.e.,*

$$g \in \bigcup_{x\in G} xTx^{-1}.$$

Observe that by as x varies over the coset of $\mathfrak{x} \in U/T$, $h_g(x)$ varies over a conjugacy class $h_g(\mathfrak{x}) \subset T$. Thus to every fixed point $\mathfrak{x}$ of $l_{g^{-1}}$ corresponds a conjugacy class $h_g(\mathfrak{x}) \subset T$.

Lemma 5. *1) Let $\mathfrak{x}$ be a fixed point of $l_{g^{-1}}$ and let $t \in h_g(\mathfrak{x})$. Then*

$$\det(\mathbb{1} - \mathrm{d}l_{g^{-1}})_{\mathfrak{x}} = \det(\mathbb{1} - \mathrm{Ad}_{U/T}(t)). \tag{15}$$

2) Further for the lifting φ_g of $l_{g^{-1}}$ to $L_\lambda = U \times_T \mathbb{C}_\lambda$ we have the relation

$$\operatorname{Tr}\varphi_g(x) = \operatorname{Tr} e^\lambda(t). \tag{16}$$

Proof. 1) Let x be an element in the coset $\mathfrak{x}$ such that

$$g^{-1}x = xt. \tag{$*$}$$

The map $L_{g^{-1}} \circ R_{t^{-1}}\colon U \to U$ defined by $u \mapsto g^{-1}ut^{-1}$ then obviously still induces the map $l_{g^{-1}}\colon U/T \to U/T$ but also keeps $x \in U$ fixed:

$$L_{g^{-1}}R_{t^{-1}}(x) = g^{-1}xt^{-1} \overset{(*)}{=} x.$$

The relation $L_{g^{-1}} \circ R_{t^{-1}} \circ L_x = L_x \circ L_t \circ R_{t^{-1}}$ implies, that under the identification $\mathrm{d}L_x \circ \mathrm{d}\pi\colon \mathfrak{u}/\mathfrak{t} \xrightarrow{\cong} T_{\mathfrak{x}}(U/T)$:

$$\mathrm{d}l_{g^{-1}}\big|_{\mathfrak{x}}(Y) = \mathrm{Ad}_{U/T}(t)(Y) = tYt^{-1},$$

where $Y \in \mathfrak{u}/\mathfrak{t} \cong T_o(U/T)$.

2) To see (16) consider a linear isomorphism $j_x\colon \mathbb{C}_\lambda \to (L_\lambda)_{\pi(x)}$ defined by $j_x(z) = [x, z]$. Hence by definition of the lifting φ_g of $l_{g^{-1}}$ to L_λ we get the following relation

$$\begin{aligned}\varphi_g \circ j_x(z) &= [gx, z] = [xx^{-1}gx, z] && ((*) \Leftrightarrow x^{-1}g = t^{-1}x^{-1})\\ &= [xt^{-1}, z] = [x, e^\lambda(t)z] = e^\lambda(t)j_x(z).\end{aligned}$$

□

Consider first the case when τ is a generator of T, i.e., that the powers of τ generate T. It follows that if $\mathfrak{x}$ is fixed ander τ, and x is in the coset $\mathfrak{x}$, i.e. $\tau^{-1}x = xt$, then for all integers n

$$x^{-1}\tau^{-n}x = t^n, \qquad (t \in h_\tau(\mathfrak{x}) \subset T)$$

Thus $\mathrm{Ad}(x^{-1})$ keeps all of T invariant, i.e., $x^{-1}Tx \subset T$ (since τ is generic in T) so that the fixed points of τ correspond percisely to the cosets of the normalizer of T modulo centralizer of T. The fixed points are therefore independet of the choise of a generator of T, and naturally form the Weyl group of U

$$W := N_U(T)/Z_U(T) = N_U(T)/T.$$

This finite group acts naturally on T by permuting the roots $\alpha \in \Phi$ and on $\widehat{T}$ Hence by (16) one obtains the formula:

$$\mathrm{Tr}(\tau \text{ on } C^\infty(X; L_\lambda)) = \sum_{\mathfrak{x} \in \mathrm{Fix}(l_{\tau^{-1}})} \frac{\mathrm{Tr}(\varphi_\tau(x))}{|\det(\mathbb{1} - \mathrm{d}l_{\tau^{-1}})|_{\mathfrak{x}}|} = \sum_{w \in W} \frac{e^{w(\lambda)}(\tau)}{|\det(\mathbb{1} - \mathrm{d}l_{\tau^{-1}})|^w}$$

From the formula (15) we have

$$\det(\mathbb{1} - \mathrm{d}l_{\tau^{-1}})|_{\mathfrak{x}} = \det(\mathbb{1} - \mathrm{Ad}_{U/T}(x^{-1}\tau^{-1}x)),$$

so that $\mathrm{d}l_{\tau^{-1}}\restriction_{\mathfrak{x}} \colon T_{\mathfrak{x}}(U/T) \to T_{\mathfrak{x}}(U/T)$ just rotates the root spaces E_α by $\alpha(\tau)$, such that by (8) we obtain

$$|\det(\mathbb{1} - \mathrm{d}l_{\tau^{-1}})||^w = \left| \prod_{\alpha \in \Phi^+} (1 - e^\alpha(\tau)) \right|^2 = |D(\tau)|^2.$$

Further by (15) we have

$$\det\nolimits_{\mathbb{C}}(\mathbb{1} - \mathrm{d}l_{\tau^{-1}})|_{\mathfrak{x}} = \det\nolimits_{\mathbb{C}}(\mathbb{1} - \mathrm{Ad}_{U/T}(x^{-1}\tau^{-1}x))$$

whence we obtain by (8) in the similar way:

$$\det\nolimits_{\mathbb{C}}(\mathbb{1} - \mathrm{d}l_{\tau^{-1}})|_{\mathfrak{x}} = \prod_{\alpha \in \Phi^-} (1 - e^\alpha)(x^{-1}\tau^{-1}x) = \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})^w(\tau).$$

Consider elliptic complex

$$0 \to \Lambda^{0,0}(L_\lambda) \xrightarrow{\overline{\partial}} \Lambda^{0,1}(L_\lambda) \to \cdot \xrightarrow{\overline{\partial}} \Lambda^{0,m}(L_\lambda) \to 0.$$

It has $\overline{\partial}^2 \equiv 0$ and hence gives rise to cohomology group $H^{0,q}(U/T; L_\lambda)$. Our group U acts naturally on $H^{0,q}(U/T; L_\lambda)$ which are by elliplicity all finite dimensional. We apply the Lefschetz principle to this complex and get:

The character of the virtual module $\sum(-1)^q H^{0,q}(U/T; L_\lambda)$
should equal that of the virtual module $\sum(-1)^q \Lambda^{0,q}(L_\lambda)$

The natural representation of T on $\mathbb{C}_\lambda \otimes \Lambda^{0,q}(\mathfrak{u}/\mathfrak{t})$ given by $\lambda \otimes \Lambda^{0,q}$ induce a representation $\Omega^{0,q} = \mathrm{Ind}_T^U(\mathrm{e}^\lambda \otimes \Lambda^{0,q})$ on $H^{0,q}(U/T; L_\lambda)$. One now obtains the relation:

$$\sum_q (-1)^q \,\mathrm{Tr}(\Omega_\lambda^{0,q}\restriction H^{0,q}(X; L_\lambda)(\tau)) = \sum_{w \in W} \left[\frac{e^\lambda}{\prod_{\alpha \in \Phi^+}(1 - e^{-\alpha})} \right]^w (\tau). \tag{17}$$

From the identity $(1 - e^{-\alpha}) = e^{-1/2\alpha}(e^{1/2\alpha} - e^{-1/2\alpha})$ it follows, that

$$\prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}) = e^{-\frac{1}{2}\sum_{\alpha > 0} \alpha} \prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2}) = e^{-\rho} D,$$

hence the right hand side of (17) is of the following form:

$$\sum_{w\in W}\left[\frac{e^{\lambda}}{\prod_{\alpha\in\Phi^+}(1-e^{-\alpha})}\right]^w = \frac{1}{\prod_{\alpha\in\Phi^+} e^{\alpha/2}-e^{-\alpha/2}}\sum_{w\in W}\operatorname{sign}(w)e^{w(\lambda+\rho)}$$

Finally the Borel-Weil theorem comes into play for th left hand side of (17), to complete the story. For a dominant weight λ all the higher terms in (17) vanishes, and $\Omega_{\lambda}^{0,0}$ turns to be the by e^{λ} induced irreducible highest weight representation $\rho_\lambda\colon U \to \mathrm{GL}(H^0(X;\mathcal{O}(L_\lambda)))$ defined by (13), such that

$$\boxed{\chi_\lambda = \operatorname{Tr}(\rho_\lambda) = \frac{1}{\prod_{\alpha\in\Phi^+} e^{\alpha/2}-e^{-\alpha/2}}\sum_{w\in W}\operatorname{sign}(w)e^{w(\lambda+\rho)}}$$

□

References

[AB68] M. F. Atiyah and R. Bott. A Lefschetz fixed point formula for elliptic complexes. II. Applications. *Ann. of Math. (2)*, 88:451–491, 1968.

[Sch97] Wilfried Schmid. Discrete series. Notes by Vernon Bolton. In *Bailey, T. N. (ed.) et al., Representation theory and automorphic forms. Proceedings of an instructional conference, Edinburgh, UK, March 17-29, 1996. Providence, RI: American Mathematical Society. Proc. Symp. Pure Math. 61, 83-113* . 1997.

[Ser01] Jean-Pierre Serre. *Complex semisimple Lie algebras*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2001. Translated from the French by G. A. Jones, Reprint of the 1987 edition.

[Wel80] R. O. Wells, Jr. *Differential analysis on complex manifolds*, volume 65 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1980.

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