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Wintersemester 2010/11

7. Übung Globale Analysis I

Abgabe am Montag, den 6. Dezember, in der Vorlesungspause.

Bei Fehlern oder Fragen bitte eine eMail an: `brief`α`fabianmeier.de` .

Question 1. (5 points)
Let $\omega \in \Lambda^1(\mathbb{R}^2 \setminus \{0\})$ be defined by

$$\omega = -\frac{x_2}{x_1^2 + x_2^2}\, dx_1 + \frac{x_1}{x_1^2 + x_2^2}\, dx_2.$$

Show that ω is closed but not exact (i.e. $d\omega = 0$, but there is no $f \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ with $\omega = df$).

Hint: Consider the integral over ω along the curve $\gamma(t) = (\cos t, \sin t)$, $0 \leq t \leq 2\pi$.

Question 2. (5 points)
Let $\omega \in \Lambda^p(M)$. Show that for all $X_0, ..., X_p \in \mathcal{V}(M)$ we have the following formula:

$$\begin{aligned} d\omega(X_0, ..., X_p) = &\sum_{i=0}^{p} (-1)^i X_i \left(\omega(X_0, ..., \hat{X}_i, ..., X_p) \right) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_p). \end{aligned}$$

A hat over a variable means that this variable is omitted.

Hint: First show that the righthand side is a $C^\infty(M)$ alternating multilinear form on $\mathcal{V}(M)^{p+1}$. Then it suffices to prove this formula for every chart (U, ϕ) of M and for $X_0, ..., X_p \in \left\{ \frac{\partial}{\partial x_i} : 0 \leq i \leq p \right\}$.

Question 3 (De Rham-Kohomologie des 2-Torus). (5 points)
Let $\mathrm{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

1. Explain why $H^0(\mathrm{T}^2) = \mathbb{R}$.
2. Let $\omega = \phi(x,y)dx + \psi(x,y)dy \in \Omega^1(\mathrm{T}^2)$, i.e. ϕ and ψ are $\mathbb{Z}^2$-periodic. Let ω be closed. Show that the functions

$$y \mapsto \int_0^1 \phi(x', y) dx', \quad x \mapsto \int_0^1 \psi(x, y') dy'$$

are constant.

3. Let ω be given as above. Prove that ω is exact if and only if

$$\int_0^1 \phi(x',0)dx' = 0 \text{ und } \int_0^1 \psi(0,y')dy' = 0.$$

Show that this implies $H^1(\mathrm{T}^2) = \mathbb{R}[dx] \oplus \mathbb{R}[dy]$.

4. Let $\omega = f(x,y)dx \wedge dy$ be a 2-form , i.e. f is $\mathbb{Z}^2$-periodic. Show that ω is exact if and only if

$$\int_0^1 \int_0^1 f(x,y)dxdy = 0.$$

Prove that this implies $H^2(\mathrm{T}^2) = \mathbb{R}[dx \wedge dy]$.

Hint for (4): Look at the form $\eta = \phi(x,y)dx + \psi(x,y)dy$ with

$$\phi(x,y) = -\int_0^y \int_0^1 f(x',y')dx'dy', \quad \psi(x,y) = \int_0^x f(x',y)dx' - x\int_0^1 f(x',y)dx'.$$

Under which circumstances does η define a smooth 1-form on T^2?

Question 4 (Pullback). (4 points)

Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the map

$$f(x,y,z) = \big(x - \sin y,\ e^{yz} - 1,\ 2x\big)$$

Calculate the pullback $f^*\big(z^2\, dx \wedge dy\big)$. The result should be of the form $g_1\, dx \wedge dy + g_2\, dy \wedge dz + g_3\, dz \wedge dx$, where g_1, g_2, g_3 are explicit functions.