Prof. Dr. Werner Müller
Dr. Fabian Meier Wintersemester 2010/11

5. Übung Globale Analysis I

Abgabe am Montag, den 22. November, in der Vorlesungspause.

Bei Fehlern oder Fragen bitte eine eMail an: `brief`α`fabianmeier.de` .

Question 1 (Dependence of $\mathcal{L}$ on vector fields). (5 points)

Give an example of smooth vector fields V_1, V_2 and W on $\mathbb{R}^2$ such that $V_1 = V_2 = \frac{\partial}{\partial x}$ along the x-axis but $L_{V_1}W \neq L_{V_2}W$ at the origin.

Remark: This shows that it is really necessary to know the vector field V to compute $(\mathcal{L}_V W)_p$; it is not enough just to know the vector V_p or even to know the values of V along an integral curve of V.

Question 2 (Lie algebras). (5 points)

Let $\mathfrak{g} \subset M(n, \mathbb{R})$ be a Lie algebra of matrices, i.e. a subvectorspace, so that

$$\forall_{A,B\in\mathfrak{g}}\, [A, B] = AB - BA \in \mathfrak{g}.$$

We define for $g \in \mathrm{GL}(n, \mathbb{R})$ the map

$$l_g\colon\ \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R}), \quad h \mapsto gh.$$

1. Let $A \in T_{\mathrm{Id}}\,\mathrm{GL}(n, \mathbb{R}) \cong M(n, \mathbb{R})$. Show

$$(l_g)_* A := (dl_g)_{\mathrm{Id}}(A) = gA.$$

2. For $A \in \mathfrak{g}$ we look at the induced left-invariant vector field

$$X_A \in C^\infty(T\,\mathrm{GL}(n, \mathbb{R})), \quad X_A(g) = gA.$$

   Prove

$$[X_A, X_B](g) = g[A, B], \quad [X_A, X_B] = X_{[A,B]}.$$

Question 3 (Flow in $O(n)$). (5 points)

Let $A \in \mathrm{Mat}_n(\mathbb{R})$ be skew-symmetric, i.e., $A^t = -A$. Show, that $\exp(tA) \in O(n)$. Let

$$f^t\colon O(n) \to O(n)$$

be defined as $f^t(B) = B\exp(tA)$. Show that f^t is a 1-parameter group of diffeomorphisms. Determine the vector field X on $O(n)$ that induces the flow f^t.

Question 4 (Satz 1.35). (4 points)

Prove the following theorem:

Let M and N be manifolds of dimension k and n and let $f : M \to N$ be an immersion. Then around every point P there exist charts $\phi : U \to B^k$ on M and $\psi : V \to B^k \times B^{n-k}$ on N (where B^j means the unit ball in dimension j), so that for all $u \in U$ we have $\psi \circ f(u) = (\phi(u), 0)$