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Wintersemester 2010/11

12. Übung Globale Analysis I

Abgabe am Montag, den 24. Januar, in der Vorlesungspause.

Bei Fehlern oder Fragen bitte eine eMail an: `brief`α`fabianmeier.de` .

Question 1 (area of a surface of revolution). (5 points)

Let $r : (0,1) \to \mathbb{R}_+$ be a bounded smooth function with bounded derivative and

$$X := \left\{(x,y,z) \in \mathbb{R}^3 \,\middle|\, x^2 + y^2 = r(z)^2,\ z \in (0,1)\right\}$$

the respective surface of revolution. It forms a Riemannian manifold whose metric is induced from $\mathbb{R}^3$. Calculate the integral of the volume form as function of r.

Question 2 (Integral theorem of Gauß (4.16)). (5 points)

Let $G \subset \mathbb{R}^n$ a domain of integration, $U \subset \mathbb{R}^n$ open with $\overline{G} \subset U$ and let $F : U \to \mathbb{R}^n$ be a C^1vector field. For $y \in \partial_r G$ let $\nu(y)$ the outer unit normal vector. Then we have

$$\int_G \mathrm{div} F(x) dx = \int_{\partial_r G} \langle F(y), \nu(y)\rangle d\mu_{\partial_r G}(y)$$

Show this with the help of Stokes' theorem.

Hint: Use the $(n-1)$-form

$$\sum_i (-1)^{i+1} F^i \, dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n,$$

where F^i are the components of F.

Question 3 (Covering space). (4 points)

Let M and N be compact Riemannian manifolds and $\pi : M \to N$ an n-fold covering. If N has a volume form α, then we get an n-form $\pi^*(\alpha)$ on M. Prove that

$$\int_M \pi^*(\alpha) = n \cdot \int_N \alpha.$$

Question 4 (fundamental group). (5 points)

Let $\pi_1^g(M;p)$ be the smooth fundamental group of a manifold M for the point $p \in M$, i.e. the set of piecewise smooth curves which start and end in p, modulo homotopy (which means homotopies which are piecewise homotopies of smooth curves.) Show that the map

$$\begin{aligned} \pi_1^g(M,p) \times H^1(M;\mathbb{R}) &\to \mathbb{R} \\ (\gamma, \alpha) &\mapsto \int_\gamma \alpha \end{aligned}$$

is well-defined.