Prof. Dr. Werner Müller
Dr. Fabian Meier

10. Übung Globale Analysis I

Abgabe am Montag, den 10. Januar, in der Vorlesungspause.

Bei Fehlern oder Fragen bitte eine eMail an: `brief`α`fabianmeier.de` .

Question 1 (Künneth theorem). (5 points)

Let $M = B \times F$ a product of the smooth compact manifolds B and F. Let π_B and π_F the projections of M onto the two factors. The bilinear map

$$\begin{aligned} f : \Omega(B) \times \Omega(F) &\to \Omega(M) \\ (\alpha, \beta) \quad &\mapsto \pi_B^*(\alpha) \wedge \pi_F^*(\beta) \end{aligned}$$

is a cochain map and induces a homomorphism

$$f_* : H(B) \otimes H(F) \to H(B \times F) = H(M).$$

Show that f_* is an isomorphism, i.e.

$$H^p(B \times F) \cong \bigoplus_{j=1}^{p} H^j(B) \otimes H^{p-j}(F).$$

Hint: Use induction with respect to the number of open sets in a good covering of B. Use the Mayer-Vietoris sequence and the five lemma for the proof of the induction step.

Question 2 (Connected sum). (6 points)

Let M_1, M_2 be smooth manifolds of dimension n. Let p_1, p_2 points with $p_i \in M_i$ and a chart W_i, whose chart map ϕ_i maps bijectively onto the open disc $B_2(0)$. We define

$$M_1 \# M_2 = \left(M_1 \backslash \phi_1^{-1}\left(\overline{B_1(0)}\right)\right) \dot{\cup} \left(M_2 \backslash \phi_2^{-1}\left(\overline{B_1(0)}\right)\right) \Big/ \sim,$$

where $\sim$ on W_1 resp. W_2 is defined as $x_1 \sim \phi_1^{-1} \circ \mu \circ \phi_2(x_2)$; the map μ on $B_2(0) \backslash \overline{B_1(0)}$ is given by $(r, \alpha) \mapsto (3 - r, \alpha)$ in polar coordinates.

1. Show that $M_1 \# M_2$ has a unique smooth structure if we require that the inclusions $\left(M_i \backslash \phi_i^{-1}\left(\overline{B_1(0)}\right)\right)$ are smooth embeddings.

2. Prove that $H^i\left(M_1 \# M_2\right) \cong H^i(M_1) \oplus H^i(M_2)$ for $0 < i < n$, if we assume the following things: The nth cohomology of M_1, M_2 and $M_1 \# M_2$ is isomorphic to $\mathbb{R}$; if you cut out a closed ball the nth cohomology vanishes. (In January we will see that these assumptions are always true if the manifold is compact and orientable).

3. Show that the # operation enables us to construct infinitely many manifolds out of a 2-torus which are not diffeomorphic to each other.

Question 3 (Hypersurface). (5 points)

Let M be connected with $H^1(M) = 0$ and let $N \subset M$ be a closed connected hypersurface, i.e. $\dim N = \dim M - 1$.

1. Prove: $M \setminus N$ has exactly two path connected components which have N as their common boundary. You may use that N has a so-called tube neighbourhood $T : T \supset N$ is open and there is a diffeomorphism $\tau\colon\ T \to N \times (-1, 1)$ with $\tau(x) = (x, 0)$ for $x \in N$.

2. Give a counterexample which shows that the condition $H^1(M) = 0$ is necessary.

Question 4 (Some lonely points in $\mathbb{R}^n$). (4 points)

Let $\mathbb{R}^n \backslash \{x_1, \ldots, x_k\}$ be the Euclidian space without k points. Compute its cohomology.