

**Inverse scattering, resonances
and
functional determinants**

Werner Müller

Mathematisches Institut
Universität Bonn

Regensburg, 23. März 2004

The purpose of this talk is to review some aspects of resonances in mathematics and physics. Resonances are defined as poles of the meromorphic continuation of the resolvent. They are related to the long time behaviour of the wave equation. In physics a resonance $E - i\gamma$ is related to a dissipative metastable state with energy E and decay rate γ . In mathematics resonances are discrete spectral data for elliptic operators on noncompact manifolds which replace eigenvalues in the compact case.

1. Closed systems and eigenvalues

To begin with we review some aspects of spectral theory for closed systems.

I) Geometry

In the geometric setting we consider a closed Riemannian manifold X with metric g and the associated Laplace operator on functions

$$\Delta : C^\infty(X) \rightarrow C^\infty(X).$$

Then Δ is essentially self-adjoint and has pure point spectrum

$$\text{Spec}(\Delta) : 0 = \lambda_0 < \lambda_1 \leq \dots \rightarrow +\infty$$

consisting of eigenvalues λ_j of finite multiplicity with eigenfunctions

$$\Delta \phi_i = \lambda_i \phi_i, \quad i \in \mathbb{N}_0.$$

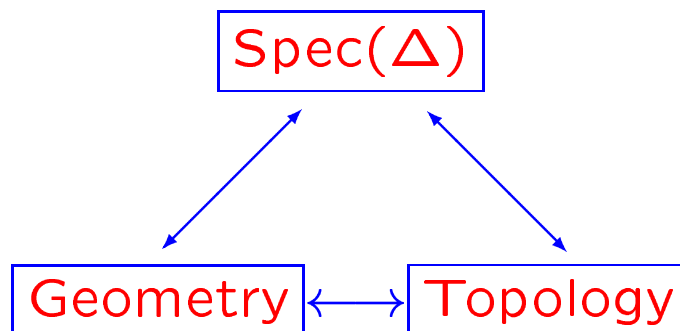
If X is compact, but has a nonempty boundary

$$\partial X \neq \emptyset,$$

we need to impose boundary conditions. Natural choices are **Dirichlet** or **Neumann** boundary conditions:

$$\phi|_{\partial X} = 0, \quad \frac{\partial \phi}{\partial \nu} \Big|_{\partial X} = 0.$$

The corresponding self-adjoint extension of Δ has again pure point spectrum. One of the main problems is to understand the relation between **Spec(Δ)** and the geometry and topology of X :



II) Quantum mechanics

In quantum mechanics we consider the Schrödinger operator

$$H = -h^2 \Delta + V, \quad V \in C^\infty(\mathbb{R}^n).$$

Assume that

$$V(x) \rightarrow \infty, \quad \|x\| \rightarrow \infty.$$

Then H has pure point spectrum for $h \ll 1$. More generally, we may consider energies near a confining nondegenerate energy level E , which means that

$$\left\{ (p, x) : \frac{1}{2}p^2 + V(x) = E \right\}$$

is a smooth bounded hypersurface in phase space.

Problems:

- inverse spectral theory
- distribution of eigenvalues and classical dynamics
- fine structure of the spectrum

Results and Methods

1. Weyl's law

i) Geometry.

Let X be a compact Riemannian manifold and let

$$0 \leq \lambda_0 \leq \lambda_1 \leq \dots \rightarrow \infty$$

be the spectrum of the Laplacian Δ , with Dirichlet or Neumann boundary conditions, if $\partial X \neq \emptyset$. Define the eigenvalue counting function by

$$N(\lambda) = \#\{i \mid \lambda_i \leq \lambda\}.$$

Let $n = \dim X$. Then we have Weyl's law:

$$N(\lambda) = \frac{\text{vol}(X)}{(4\pi)^{n/2} \Gamma(\frac{n}{2} + 1)} \lambda^{n/2} + o(\lambda^{(n-1)/2})$$

as $\lambda \rightarrow \infty$.

- If the geodesic flow is ergodic the remainder term satisfies

$$R(\lambda) = o(\lambda^{(n-1)/2})$$

as $\lambda \rightarrow \infty$.

ii) Quantum mechanics.

$$H = -h^2 \Delta + V, \quad V \in C^\infty(\mathbb{R}^n).$$

Eigenvalues E of H can be counted using only the classical Hamiltonian

$$H(x, p) = \frac{1}{2} \|p\|^2 + V(x)$$

and Planck's constant h . Let $E_0 < E_1$ and suppose that for $E \in [E_0, E_1]$

$$\{(x, p) : H(x, p) = E\}$$

is a bounded hypersurface. Then

$$\begin{aligned} & \# \{E : E \in [E_0, E_1]\} \\ & \sim \frac{\text{vol}(\{(x, p) : E_0 \leq H(x, p) \leq E_1\})}{(2\pi h)^n} \end{aligned}$$

2. Heat equation method

There is an asymptotic expansion

$$\mathrm{Tr}(e^{-t\Delta}) = \sum_j e^{-t\lambda_j} \underset{t \rightarrow 0^+}{\sim} (4\pi t)^{-n/2} \sum_{k=0}^{\infty} a_k t^k$$

- The heat coefficients a_j 's are locally determined by the metric.

$$a_0 = \mathrm{Vol}(X), \quad a_1 = \frac{1}{6} \int_X R(x) \mathrm{dvol}, \quad \dots$$

Here $R(x)$ denotes the scalar curvature.

- Similar formula for the Schrödinger operator

3. Wave equation method

The study of the wave equation leads to relation between $\mathrm{Spec}(\Delta)$ and the length spectrum

$$\mathrm{Lsp}(X, g) := \{l(\gamma) \mid \gamma \text{ closed geodesic}\}$$

Let

$$\hat{\mu} = \mathrm{Tr}(e^{it\sqrt{\Delta}})$$

Then

sing supp $\hat{\mu} \subseteq$ length spectrum

- “=”, if g is generic
- Semi-classical trace formula for Schrödinger operators

4. Higher spectral invariants

Regularized determinants are important spectral invariants of elliptic operators. Let

$$\zeta_{\Delta}(s) = \sum_{\lambda_j > 0} \lambda_j^{-s}, \quad \operatorname{Re}(s) > \frac{\dim X}{2}.$$

The regularized determinant of Δ is defined by

$$\det \Delta = \exp \left(-\frac{d}{ds} \zeta_{\Delta}(s) \Big|_{s=0} \right).$$

2. Open systems and resonances

Consider a noncompact Riemannian manifold X and

$$\Delta: C^\infty(X) \rightarrow C^\infty(X)$$

or

$$H = -h^2 \Delta + V, \quad V \in C_c^\infty(\mathbb{R}^n).$$

New features:

- particles may escape to infinity
- there may be no bound states

Example: Obstacle scattering in \mathbb{R}^n .

Possible replacement for eigenvalues: **Resonances**

- Resonances may be defined as poles of the scattering matrix

More general: Consider a self-adjoint operator

$$H : \mathcal{D} \rightarrow \mathcal{H}$$

in a Hilbert space \mathcal{H} . Let

$$R(z) = (H - z)^{-1} : \mathcal{H} \rightarrow \mathcal{H}$$

be the resolvent, which is bounded in the half-plane $\text{Im}(z) < 0$.

Assumption: Suppose there exist

$$\mathcal{H}_{cp} \subset \mathcal{H} \subset \mathcal{H}_{loc}$$

such that

$$R(z) : \mathcal{H}_{cp} \rightarrow \mathcal{H}_{loc}$$

has meromorphic extension to a Riemann surface $\Sigma \rightarrow \mathbb{C}$ covering \mathbb{C} . Then define the **resonance set** as

$$\text{Res}(H) = \{\eta \in \Sigma \mid \eta \text{ pole of } R(z)\}.$$

- The point spectrum is included in $\text{Res}(H)$. Usually resonances have non-zero imaginary parts and a distinction should be made.

Example:

$$H = -\Delta + V, \quad V \in C_c^\infty(\mathbb{R}^n)$$

Let $\Lambda \rightarrow \mathbb{C}$ be the logarithmic covering. Then

$$(H - z^2)^{-1} : L_{cpt}^2(\mathbb{R}^n) \rightarrow L_{loc}^2(\mathbb{R}^n)$$

has a meromorphic extension to

$$\mathbb{C}, \text{ if, } n = 2k + 1; \quad \Lambda, \text{ if, } n = 2k.$$

- In many settings a meromorphic extension of the scattering matrix exists and resonances can be defined as poles of the scattering matrix, denoted by $\text{Res}_{sc}(H)$.

Resonances describe the longtime behaviour of solutions of the wave equation.

Let

$$H = -h^2 \Delta + V, \quad V \in C_c^\infty(\mathbb{R}^n),$$

and assume that $V \geq 0$. Let

$$\left(-\partial_t^2 - H\right) u = 0; \quad u|_{t=0}, \partial_t u|_{t=0} \in C_c^\infty.$$

Then

$$u(t, x) \sim \sum_{\lambda \in \text{Res}(H)} e^{it\lambda} v_\lambda(x)$$

for $x \in K$, $K \subset \mathbb{R}^n$ any compact set, and $t \rightarrow \infty$.

$\text{Re}(\lambda) \longleftrightarrow$ rate of oscillation

$\text{Im}(\lambda) \longleftrightarrow$ rate of decay

- Quantum resonances $\lambda = E_0 - i\Gamma$ describe states which have an initial energy E_0 and which decay at an exponential rate Γ .
- Resonances replace bound states in any system in which particles have the possibility to escape to infinity.
- At high energy the density of resonances increases and their distribution is sensitive to the properties of the classical flow.

General problems

- **Analytic continuation of the resolvent:** For what manifolds or perturbations exists a meromorphic extension of $R(z) = (\Delta - z)^{-1}$?
- **Resonance counting:** Upper bounds for the number of resonances, Weyl's law?
- **Existence of resonances:** Lower bounds for the number of resonances
- **Distribution of resonances.**
- **Semiclassical analysis and trace formulas.**
- **Inverse problem:** What information about the scatterer can be obtained from the resonances?
- **Perturbation theory of resonances.**

Some results

1. Upper bounds: Let

$$N_{\text{res}}(r) = \#\{\rho \in \text{Res}(H) : |\rho| \leq r\},$$

where poles are counted with multiplicities. For **obstacle scattering** and n odd, Melrose proved the optimal bound

$$N_{\text{res}}(r) \leq Cr^n.$$

2. Potential scattering

Let

$$H = -h^2 \Delta + V$$

and assume that the classical flow

$$(x, p) \longmapsto \phi^t(x, p),$$

associated to the Hamiltonian

$$H(x, p) = \frac{1}{2} \|p\|^2 + V(x),$$

is **hyperbolic**. Define the **trapped set** of the flow by

$$K_\delta(E_0) = \{(x, p) : |H(x, p) - E_0| \leq 2\delta \\ \phi^t(x, p) \not\rightarrow \infty, t \rightarrow \pm\infty\} \\ \in \mathbb{R}^{2n}.$$

Theorem(Sjöstrand).

$$\#\{E - i\Gamma : |E - E_0| < \delta, \quad \Gamma < Ch\} \\ \approx h^{-d(E_0, \delta)}.$$

Here $d(E_0, \delta)$ is given in terms of the Minkowski dimension of $K_\delta(E_0)$.

- **Further results:** Melrose, Petkov, Sjöstrand, Vojtechovský, Zworski,...

Geometric results

Riemann surfaces

Let (X, g) be a surface with a complete metric g .

Assume: X has a decomposition of the following form:

$$X = X_0 \sqcup C_1 \sqcup \cdots \sqcup C_l \sqcup Y_1 \sqcup \cdots \sqcup Y_m$$

with

1) X_0 is a compact surface with boundary

2) $C_i \cong [a_i, \infty) \times (\mathbb{R}/h_i\mathbb{Z})$, $a_i > 0$, $h_i > 0$, and

$$g|_{C_i} \cong dr^2 + e^{-2r} d\theta^2.$$

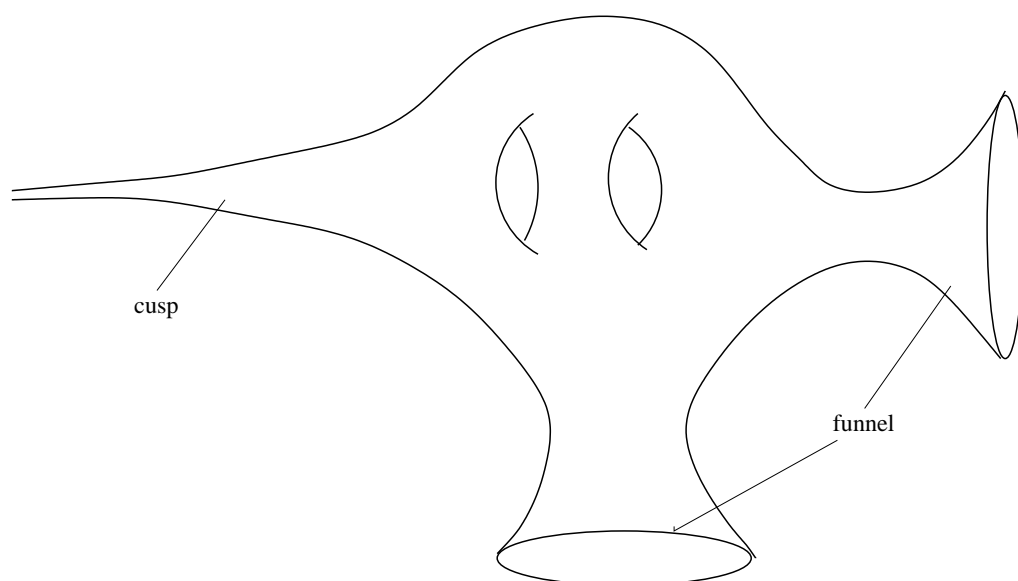
Each C_i is called a **cuspidal**.

3) $Y_j \cong [b_j, \infty) \times (\mathbb{R}/l_j\mathbb{Z})$, $b_j > 0$, $l_j > 0$, and

$$g|_{Y_j} \cong dr^2 + \cosh^2(r) d\theta^2.$$

Each Y_j is called a **funnel**. It is a subset of the **hyperbolic half-cylinder**

$$Y_j^0 = [0, \infty) \times S^1, \quad g_0 = dr^2 + \cosh^2(r) d\theta^2.$$



Let Δ be the Laplacian of X with respect to the metric g . Then

$$\Delta : C_0^\infty(X) \rightarrow L^2(X)$$

is essentially self-adjoint.

Theorem (Guillopé-Zworski)

$$R_X(s) = (\Delta - s(1 - s))^{-1} : L^2(X) \rightarrow H^2(X)$$

$$\operatorname{Re}(s) > 1/2, \quad s(1 - s) \notin \operatorname{Spec}_{pp}(\Delta)$$

extends to a meromorphic family of bounded operator

$$R_X(s) : L_{\text{cpt}}^2(X) \rightarrow H_{\text{loc}}^2(X)$$

with poles of finite rank.

Let

$$\operatorname{Res}(\Delta) = \{\eta \in \mathbb{C} : \eta \text{ pole of } R_X(s)\}.$$

For a given pole s_0 let

$$R_X(s) = \sum_{j=1}^k \frac{A_j}{(s - s_0)^j} + A_0(s)$$

be the Laurent series at s_0 .

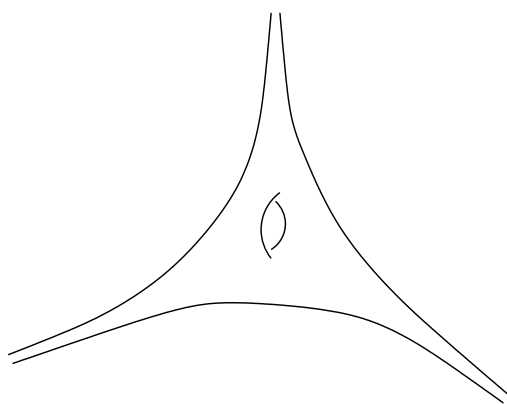
Define the multiplicity of the pole by

$$m_{s_0}(R_X) = \dim \left(\sum_{j=1}^k A_j(L_{\text{cpt}}^2(X)) \right).$$

a) Finite area surfaces

Assume: $\text{Vol}(X) < \infty$.

Then X has no funnels.



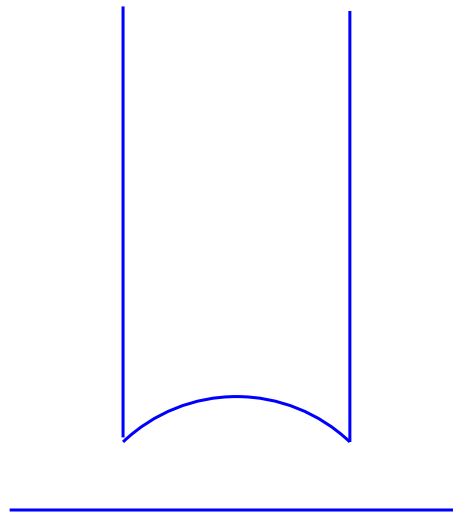
Examples: Let H be the upper half-plane equipped with the hyperbolic metric.

1) Let $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ and

$$X = \Gamma \backslash H$$

the modular surface.

The fundamental domain of $\mathrm{SL}(2, \mathbb{Z})$ is the following domain in H .



2) For $N \in \mathbb{N}$ set

$$\Gamma(N) = \{\gamma \in \mathrm{SL}(2, \mathbb{Z}) : \gamma \equiv \mathrm{Id} \pmod{N}\}$$

and $X = \Gamma(N) \backslash H$.

3) Set $X = \mathbb{R} \times S^1$. Let $f \in C^\infty(\mathbb{R}, (0, \infty))$ be such that

$$f(y) = \exp(-|y|/2), \quad |y| \gg 1.$$

Define the metric on X by

$$g = dy^2 + f(y)^4 d\theta^2.$$

Then (X, g) is complete, has finite area and $K_g \equiv -1$ near infinity.

Theorem (M., Selberg) The spectrum of Δ has the following structure:

- 1) $\text{Spec}(\Delta) = \text{Spec}_{pp}(\Delta) \cup \text{Spec}_{ac}(\Delta)$;
- 2) $\text{Spec}_{pp}(\Delta) : 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$;
- 3) $\text{Spec}_{ac}(\Delta) = [1/4, \infty)$.

- embedded eigenvalues may exist
- The case $X = \Gamma \backslash H$ was first treated by Selberg, 1954.
- For $X = \text{SL}(2, \mathbb{Z}) \backslash H$ scattering resonances are related to the zeros of the Riemann zeta function:

$$\text{Res}_{sc}(\Delta) = \{\rho \in \mathbb{C} : \zeta(2\rho) = 0, 0 < \text{Re}(\rho) < 1/2\}.$$

Let

$$N_{\text{dis}}(\lambda) = \#\{i: \lambda_i \leq \lambda^2\}$$

be the eigenvalue counting function and let

$$N_{\text{res}}(\lambda) = \sum_{\substack{\rho \in \text{Res}_{\text{sc}}(\Delta) \\ |\rho| \leq \lambda}} m(\rho)$$

be the counting function of the scattering resonances.

Theorem (M., Parnowski). For all $\epsilon > 0$

$$N_{\text{dis}}(\lambda) + \frac{1}{2}N_{\text{res}}(\lambda) = \frac{\text{Vol}(X)}{4\pi} \lambda^2 + O(\lambda^{3/2+\epsilon})$$

as $\lambda \rightarrow \infty$.

- The proof is based on the wave equation method

Let

$$n(T) = \#\{\rho = \beta + i\gamma \mid \zeta(\rho) = 0, \\ 0 < \beta < 1, \quad |\gamma| \leq T\}.$$

be the counting function of the Riemann zeros.

Riemann-von Mangoldt formula

$$n(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$$

This implies

Theorem (Selberg) For $X = \mathrm{SL}(2, \mathbb{Z}) \backslash H$

$$N_{\mathrm{dis}}(\lambda) = \frac{\mathrm{Vol}(X)}{4\pi} \lambda^2 + O(\lambda \log \lambda)$$

as $\lambda \rightarrow \infty$.

- The existence of eigenvalues is very subtle. Eigenfunctions are believed to be related to number theory. A conjecture of Osgood, Phillips and Sarnak states that for a generic hyperbolic surface of finite area there exist only finitely many eigenvalues, which all lie below the continuous spectrum.

This is a theorem, if we allow more general metrics.

Theorem (Colin de Verdiere) For a generic surface (X, g) of finite area, Δ has no embedded eigenvalues.

Our knowledge about the counting functions is as follows

X	generic	hyberbolic	arithmetic
$N_{\text{dis}}(\lambda)$	$O(1)$?	$O(\lambda^2)$
$N_{\text{res}}(\lambda)$	$O(\lambda^2)$?	$O(\lambda \log \lambda)$

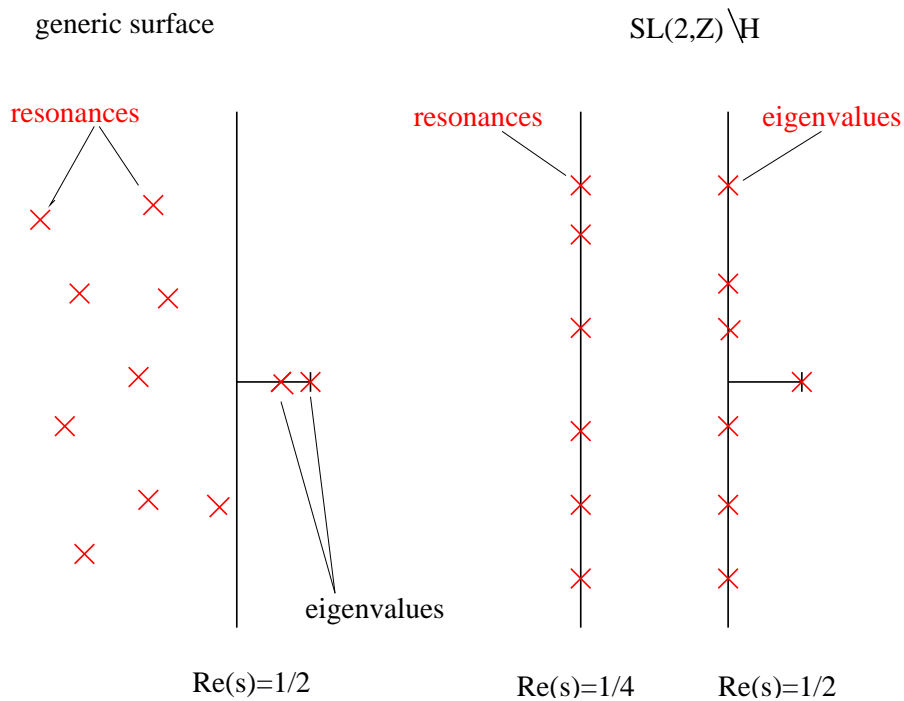
Put $\lambda = s(1 - s)$, $s \in \mathbb{C}$. Then eigenvalues λ_i correspond to points

$$s_i \in \{s : \text{Re}(s) = 1/2\} \cup [1/2, 1].$$

Scattering resonances are contained in

$$\{s : \text{Re}(s) < 1/2\} \cup [1/2, 1].$$

- **Generic surface:** Resonances are randomly distributed in $\text{Re}(s) < 1/2$.
- $\text{SL}(2, \mathbb{Z}) \backslash H$: Points s_i such that $\lambda_i = s_i(1 - s_i) \neq 1$ is an eigenvalue, are distributed on the line $\text{Re}(s) = 1/2$. If the Riemann hypothesis is true, all scattering poles are distributed on the line $\text{Re}(s) = 1/4$.



Problems:

- dynamics of resonances
- extremal principle?
- role of arithmetic surfaces

b) $\text{Vol}(X) = \infty$.

Assume that $X = \Gamma \backslash H$ is noncompact and has no cusps. It is convenient to set

$$H = -\Delta + \frac{1}{4}.$$

Theorem (Zworski)

$$\#\{E - i\Gamma : 1 < E < \lambda, \Gamma < C\} \leq C_1 \lambda^{1+\delta(\Gamma)},$$

where $\delta(\Gamma)$ is the dimension of the limit set of Γ .

- The dimension of the trapped set of the Hamiltonian flow in T^*X is equal to $2(1 + \delta(\Gamma))$.

3. Regualrized determinants

- To define functional determinants for elliptic operators on noncompact manifolds, further regularization is necessary.

Example.

$$H = -\Delta + V, \quad V \in C_c^\infty(\mathbb{R}^n)$$
$$H_0 = -\Delta.$$

The following fact are known:

- 1) $e^{-tH} - e^{-tH_0}$ is trace class for $t > 0$;
- 2) There exists an asymptotic expansion

$$\mathrm{Tr} \left(e^{-tH} - e^{-tH_0} \right) \underset{t \rightarrow 0^+}{\sim} (4\pi t)^{-n/2} \sum_{j=1}^{\infty} \alpha_{j,n} t^j.$$

3) Let $S(\lambda)$ be the on-shell scattering matrix of (H, H_0) . The following trace formula holds.

$$\begin{aligned} \mathrm{Tr} \left(e^{-tH} - e^{-tH_0} \right) &= \sum_{i=1}^N e^{-t\lambda_i} + \frac{\log \det S(0)}{2\pi i} \\ &+ \frac{1}{2\pi i} \int_0^{\infty} e^{-t\lambda} \frac{d}{d\lambda} \log \det S(\lambda) d\lambda. \end{aligned}$$

Set

$$s(\lambda) := \frac{1}{2\pi i} \log \det S(\lambda).$$

If n is odd, there exists $\epsilon > 0$ and a convergent power series expansion:

$$\frac{ds}{d\lambda}(\lambda) = \sum_{j=0}^{\infty} b_j \lambda^{j/2}, \quad 0 \leq \lambda \leq \epsilon.$$

Using these facts, we can define partial zeta functions.

$$\zeta_1(s, H, H_0) = \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \text{Tr}(e^{-tH} - e^{-tH_0}) dt, \quad \text{Re}(s) \gg 0.$$

$$\zeta_2(s, H, H_0) = \frac{1}{\Gamma(s)} \int_1^{\infty} t^{s-1} \text{Tr}(e^{-tH} - e^{-tH_0}) dt, \quad \text{Re}(s) < \beta_0.$$

- For n odd, $\zeta_1(s, H, H_0)$ and $\zeta_2(s, H, H_0)$ admit meromorphic extensions to \mathbb{C} .

Put

$$\zeta(s, H, H_0) = \zeta_1(s, H, H_0) + \zeta_2(s, H, H_0).$$

- $\zeta(s, H, H_0)$ is holomorphic at $s = 0$.

The regularized determinant can be defined as

$$\det(H, H_0) := \exp \left(-\frac{d}{ds} \zeta(s, H, H_0) \Big|_{s=0} \right).$$

This works in the following cases:

- a) Obstacle scattering in \mathbb{R}^n , n odd.
- b) Surfaces (X, g) with cusps.
- c) Hyperbolic manifolds $\Gamma \backslash H^n$, $\text{vol}(\Gamma \backslash H^n) < \infty$.
- d) Manifolds with cylindrical ends.

Let X be a surface with cusps. Then we can define a zeta function using resonances:

$$\zeta_{\Delta}(s) := \sum_{\substack{\eta \in \text{Res}(\Delta) \\ n \neq 1}} (1 - \eta)^{-s}, \quad \text{Re}(s) > 2.$$

- $\zeta_{\Delta}(s)$ admits meromorphic extension to \mathbb{C} , regular at $s = 0$.

In this case we define the regularized determinant of Δ by

$$\det \Delta := \exp \left(-\frac{d}{ds} \zeta_{\Delta}(s) \Big|_{s=0} \right).$$

- A similar construction works for hyperbolic manifolds of finite volume.

4) Inverse scattering

Let X_1, X_2 be compact Riemannian manifolds and let $N_i(\lambda)$, $i = 1, 2$, be the eigenvalue counting functions. Then X_1, X_2 are said to be **isospectral**, if

$$N_1(\lambda) = N_2(\lambda), \quad \lambda \geq 0.$$

Let (X, g) be a noncompact complete Riemannian manifold. Then eigenvalues are replaced by resonances. The corresponding question is:

- To what extent is (X, g) determined by $\text{Res}(\Delta_X)$?

We discuss some examples.

1. Obstacle scattering in \mathbb{R}^2 (Hassell-Zelditch)

Let \mathcal{O} be a bounded domain in \mathbb{R}^2 and let

$$\Omega = \mathbb{R}^2 - \mathcal{O}.$$

Let $S_\Omega(\lambda)$ be the on-shell scattering matrix for the Dirichlet problem on Ω . Set

$$s(\lambda) = -i \log \det S_\Omega(\lambda).$$

Let Ω_1, Ω_2 be two exterior domains in \mathbb{R}^2 . Then Ω_1, Ω_2 are said to be **isophasal**, if

$$s_{\Omega_1}(\lambda) = s_{\Omega_2}(\lambda), \quad \lambda \in \mathbb{R}.$$

Theorem (Hassell-Zelditch) Each class of isophasal domains is sequentially compact in the C^∞ -topology.

- The proof uses the relative determinant $\det(\Delta_\Omega, \Delta_0)$, where Δ_Ω is the Laplacian of the domain Ω with Dirichlet boundary conditions and Δ_0 is the Laplacian of \mathbb{R}^2 .

2. Hyperbolic surfaces We consider the class of complete surfaces with constant Gaussian curvature.

Theorem (M.) Let $X = \Gamma \backslash H$, $\text{Vol}(X) < \infty$. Then $\text{Res}(\Delta_X)$ determines X up to finitely many possibilities in the moduli space of the surface.

Theorem (Bordwick, Judge, Perry) Let $X = \Gamma \backslash H$ be geometrically finite, $\text{vol}(\Gamma \backslash H) = \infty$. Then $\text{Res}(\Delta_X)$ determines X up to finitely many possibilities.