Harmonic analysis on locally symmetric spaces and number theory

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Introduction

Harmonic analysis on locally symmetric spaces $\Gamma\backslash G/K$ of finite volume is closely related to the modern theory of automorphic forms and has deep connections to number theory. Of particular interest are quotients of symmetric spaces G/K by arithmetic groups Γ. In many cases, these quotients are non-compact, which implies that the Laplace operator has a large continuous spectrum. This is where scattering theory comes into play. The study of the continuous spectrum has important consequences for the theory of automorphic forms. On the other hand, the arithmetic nature of the underlying spaces has great influence on the structure of the continuous spectrum and the distribution of resonances. The purpose of this talk is to discuss some aspects of this relation between harmonic analysis and number theory.

General set up

I. Automorphic forms

i) Symmetric spaces

- G semisimple real Lie group with finite center of non-compact type
- $K \subset G$ maximal compact subgroup
- $S = G/K$ Riemannian symmetric space of non-positive curvature, equipped with G-invariant Riemannian metric g.
- geodesic inversion about any $x \in S$ is a global isometry.

Examples. 1. $\mathbb{H}^n$ hyperbolic n-space

$$\mathbb{H}^n = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon x_n > 0\right\} \cong \mathrm{SO}_0(n,1)/\,\mathrm{SO}(n).$$

The invariant metric is given by

$$ds^2 = \frac{dx_1^2 + \cdots + dx_n^2}{x_n^2}.$$

1. General set up

2. S space of positive definite $n \times n$-matrices of determinant 1.

$$S = \left\{ Y \in \mathrm{Mat}_n(\mathbb{R}) \colon \ Y = Y^*, \ Y > 0, \ \det Y = 1 \right\} \cong \mathrm{SL}(n, \mathbb{R}) / \mathrm{SO}(n)$$

- Invariant metric: $\quad ds^2 = \mathrm{Tr}(Y^{-1} dY \cdot Y^{-1} dY)$.
- $G = \mathrm{SL}(n, \mathbb{R})$ acts on S by $Y \mapsto g^t Y g$, $g \in G$.

Locally symmetric spaces

ii) Locally symmetric spaces

- $\Gamma \subset G$ a lattice, i.e., Γ is a discrete subgroup of G and $\mathrm{vol}(\Gamma \backslash G) < \infty$
- Γ acts properly discontinuously on S.
- $X = \Gamma \backslash S = \Gamma \backslash G/K$ locally symmetric space

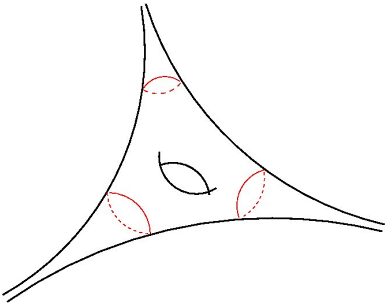
Example: $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2) \cong \mathbb{H}^2 = \{z \in \mathbb{C}\colon\ \mathrm{Im}(z) > 0\}$.
For $N \in \mathbb{N}$ let

$$\Gamma(N) = \left\{\gamma \in \mathrm{SL}(2,\mathbb{Z})\colon \gamma \equiv \mathrm{Id} \mod N\right\}.$$

- principal congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$ of level N. It acts on $\mathbb{H}^2$ by fractional linear transformations:

$$\gamma(z) = \frac{az+b}{cz+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N).$$

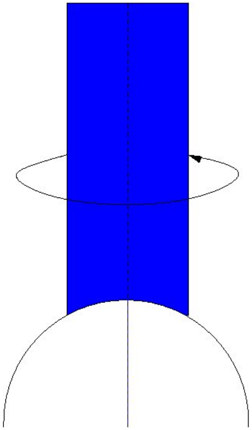
- $\Gamma(N)\backslash\mathbb{H}^2$ Riemann surface, non-compact, $\mathrm{Area}(\Gamma(N)\backslash\mathbb{H}^2) < \infty$.



A hyperbolic surface with 3 cusps.

Fundamental domain

A discrete group can be visualized by its fundamental domain.



The standard fundamental domain of the modular group $\mathrm{SL}(2, \mathbb{Z})$.

Harmonic analysis

iii) Harmonic analysis

- $\mathcal{D}(S)$:= ring of G–invariant differential operators on S, i.e.,

$$D\colon C^\infty(S) \to C^\infty(S), \quad D \circ L_g = L_g \circ D, \ \forall g \in G,$$

where $L_g f(x) = f(gx)$, $g \in G$.

$$\Delta = -\operatorname{div} \circ \operatorname{grad} = -\frac{1}{\sqrt{\det(g_{ij})}} \sum_{i,j} \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{\det(g_{ij})} \frac{\partial}{\partial x_j} \right)$$

- Laplace operator of S, $\Delta \in \mathcal{D}(S)$.

Harish-Chandra: $\mathcal{D}(S)$ is commutative and finitely generated, $r = \mathrm{rank}(S)$ minimal number of generators.

- $D \in \mathcal{D}(S) \Rightarrow D^* \in \mathcal{D}(S)$, $\mathcal{D}(S)$ commutative algebra of normal operators.

Problem: Study the spectral resolution of $\mathcal{D}(S)$ in $L^2(\Gamma \backslash S)$.

Automorphic forms

Definition: $\phi \in C^\infty(S)$ is called automorphic form, if it satisfies the following conditions:

1) $\phi(\gamma x) = \phi(x)$, $\gamma \in \Gamma$;
2) $D\phi = \lambda_D \phi, \ \forall D \in \mathcal{D}(S)$;
3) ϕ is of moderate growth.

- Every joint eigenfunction of $\mathcal{D}(S)$ which is in $L^2(\Gamma \backslash S)$, is an automorphic form.

Example: $X = \Gamma\backslash\mathbb{H}^2$, $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ lattice.

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right), \quad z = x + iy,$$

hyperbolic Laplace operator, $\mathcal{D}(\mathbb{H}^2) = \mathbb{C}[\Delta]$.

Harmonic analysis

Maass automorphic form: $\phi \in C^\infty(\mathbb{H}^2)$ such that

1) $\phi(\gamma z) = \phi(z), \quad \gamma \in \Gamma$;
2) $\Delta\phi = \lambda\phi$;
3) $\int_X |\phi(z)|^2 \, dA(z) < \infty$.

Let $\bar{\Delta}$ be the closure of $\Delta \colon C_c^\infty(\Gamma\backslash\mathbb{H}^2) \to L^2(\Gamma\backslash\mathbb{H}^2)$. Solutions λ of 1) – 3) are in the point spectrum of $\bar{\Delta}$.

Theorem(Selberg, 1954): $L^2(\Gamma\backslash\mathbb{H}^2) = L^2_{pp}(\Gamma\backslash\mathbb{H}^2) \oplus L^2_{ac}(\Gamma\backslash\mathbb{H}^2)$,

1) $\sigma_{pp}(\bar{\Delta})\colon 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$, eigenvalues of finite multiplicities,
2) $\sigma_{ac}(\bar{\Delta}) = [1/4, \infty)$,
3) $L^2_{ac}(\Gamma\backslash\mathbb{H}^2)$ can be described in terms of Eisenstein series.

Thus all eigenvalues $\lambda_i \geq 1/4$ are embedded into the continuous spectrum. This makes it very difficult to study these eigenvalues.

Applications

Why should we study automorphic forms?

1) Automorphic L-functions – Langlands program

Example: Let $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ and $F \subset \mathbb{H}^2$ the standard fundamental domain. Let $f \in C^\infty(\mathbb{H}^2)$ be a square integrable Γ-automorphic form with eigenvalue $\lambda = 1/4 + r^2$, $r \geq 0$, i.e., f satisfies

$$f(\gamma z) = f(z),\ \gamma \in \Gamma, \quad \Delta f = (1/4 + r^2) f, \quad \int_F |f(z)|^2 dA(z) < \infty.$$

In addition assume that f is symmetric w.r.t. to the reflection $x + iy \mapsto -x + iy$.
Since f satisfies $f(z+1) = f(z)$ it admits the following Fourier expansion w.r.t. to x:

$$f(x+iy) = \sum_{n=1}^{\infty} a_n y^{1/2} K_{ir}(2\pi n y) \cos(2\pi n x),$$

where

Automorphic L-functions

$$K_\nu(y) = \int_0^\infty e^{-y\cosh t}\cosh(\nu t)\,dt$$

is the modified Bessel function. Let

$$L(s,f) := \sum_{n=1}^\infty \frac{a_n}{n^s}, \quad \mathrm{Re}(s) > 1.$$

The modularity of f implies that $L(s,f)$ admits a meromorphic extension to $\mathbb{C}$, and satisfies a functional equation. Let

$$\Lambda(s,f) = \pi^{-s}\Gamma\left(\frac{s+ir}{2}\right)\Gamma\left(\frac{s-ir}{2}\right)L(s,f).$$

Then the functional equation is $\Lambda(s) = \Lambda(1-s)$.

- $L(s,f)$ is an example of an **automorphic L-function**.
- This construction can be generalized to automorphic forms w.r.t. other semisimple (or reductive) groups.

Automorphic L-functions

Basic problem: Establish analytic continuation and functional equation in the general case

Langlands' functoriallity principle: Provides relation between auotmorphic forms on different groups by relating the corresponding L-functions.

Basic conjecture: All L-functions occurring in number theory and algebraic geometry are automorphic L-functions.

- Leads to deep connections between harmonic analysis and number theory.

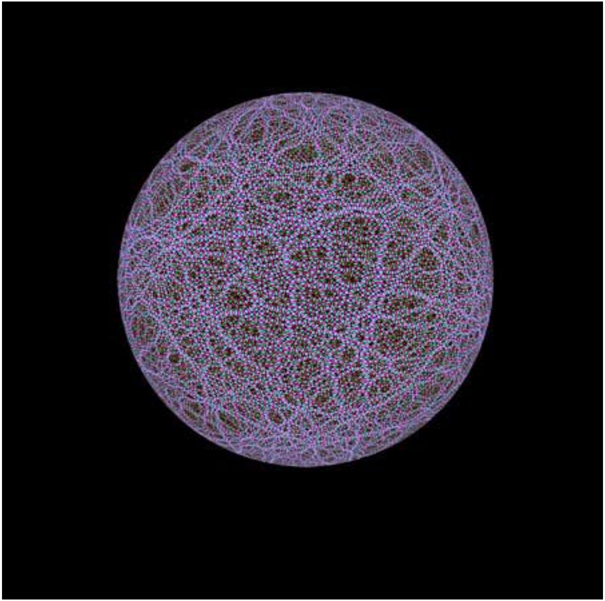
Example. A. Wiles, proof of the Shimura-Taniyama conjecture: The L-function of an elliptic curve is automorphic.

Langlands program: This theorem holds in much greater generality. There is a conjectured correspondence

$$\left\{ \begin{matrix} irreducible \quad n-dim. \\ repr's \quad of \quad \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \end{matrix} \right\} \rightarrow \left\{ \begin{matrix} automorphic \quad forms \\ of \quad \mathrm{GL}(n) \end{matrix} \right\}$$

Mathematical physics

- $\Gamma\backslash\mathbb{H}^2$ surfaces negative curvature, geodesic flow is ergodic.
- $\Gamma\backslash\mathbb{H}^2$ models for quantun chaos
- L^p-estimates for eigenfunctions, "random wave conjecture"
- "Quantum unique ergodicity"



Superposition of random waves on the sphere, Eric Heller

For $N \in \mathbb{N}$ let

$$\Gamma(N) = \{\gamma \in \mathrm{SL}(2, \mathbb{Z})\colon \gamma \equiv \mathrm{Id} \mod N\}.$$

Let $X(N) = \Gamma(N)\backslash\mathbb{H}^2$. Let $\Delta\phi_j = \lambda_j\phi_j$, $\{\phi_j\}_{j\in\mathbb{N}}$ an orthonormal basis of $L^2_{pp}(X(N))$. Existence: see Theorem 5.

L^∞-conjecture: Fix $K \subset X(N)$ compact. For $\varepsilon > 0$

$$\| \phi_j|_K \|_\infty \ll_\varepsilon \lambda_j^\varepsilon, \quad j \in \mathbb{N}.$$

- Implies Lindelöf hypothesis for $\zeta(s)$, and also for $L(s, \phi_j)$.

$$L(1/2 + it, \phi) \ll_\varepsilon (C(\phi, t))^\varepsilon.$$

where $C(\phi, t)$ is the analytic conductor of ϕ. For a Dirichlet L-function $L(s, \chi)$, $C(\chi, t) = (|t| + 1)(q + 1)$.
Seger, Sogge: L^∞ bounds on general compact surfaces.

$$\| \phi_j \|_\infty \ll \lambda_j^{1/4}.$$

Mathematical physics

Theorem (Iwaniec-Sarnak, 1995): ϕ_j on $X(N)$.

$$\| \phi_j \|_\infty \ll \lambda_j^{5/24}.$$

Let

$$\mu_j = |\phi_j(z)|^2 \, dA(z).$$

μ_j is a probability measure on $X(N)$.
Quantum unique ergodicity conjecture:

$$\mu_j \to \frac{1}{\mathrm{Area}(X(N))} dA(z), \quad j \to \infty.$$

The existence of infinitely many L^2 eigenfunctions of Δ on $X(N)$ is essential for these conjectures.

II. Scattering theory

The existence of L^2-automorphic forms is intimately related with the structure of the continuous spectrum. The continuous spectrum is described by Eisenstein series.

Stationary approach Selberg, Faddejev, Pawlow, ..., Langlands

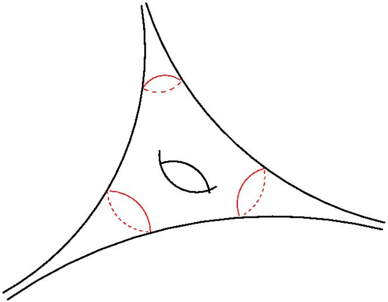
i) Rank 1

Let $X = \Gamma\backslash\mathbb{H}^2$, where $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ is a lattice. Then X has the following structure

$$X = X_0 \sqcup Y_1 \sqcup \cdots \sqcup Y_m,$$

where X_0 is a compact surface with boundary and

$$Y_j \cong [a_j, \infty) \times S^1, \quad g|_{Y_j} \cong du^2 + e^{-2u} d\theta^2,\ j = 1, ..., m.$$



A hyperbolic surface with 3 cusps.

The surface X can be compactified by adding m points $a_1, \ldots, a_m$:

$$\overline{X} = X \cup \{a_1, \ldots, a_m\}.$$

- $\overline{X}$ is a closed Riemann surface.
- The points $a_1, \ldots, a_m$ are called cusps. They correspond to parabolic fixed points $p_1, \ldots, p_m \in \mathbb{R} \cup \{\infty\}$ of Γ.
- Each cusp a_k has an associated generalized eigenfunction which is explicitely contructed as Eisenstein series.

Eisenstein series

Example: $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. Then $\Gamma\backslash\mathbb{H}^2$ has a single cusp ∞. Recall that on $\mathbb{H}^2$ the Laplace operator is given by

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

Thus, for $s \in \mathbb{C}$, we have $\Delta y^s = s(1-s)y^s$. Let

$$\Gamma_\infty = \{\gamma \in \Gamma \colon \gamma(\infty) = \infty\} = \left\{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z}\right\}.$$

Since $\mathrm{Im}(z+n) = \mathrm{Im}(z)$, $n \in \mathbb{Z}$, the function y^s is invariant under Γ_∞. To get a Γ-invariant function, we need to average over Γ:

$$E(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma z)^s = \sum_{(m,n)=1} \frac{y^s}{|mz+n|^{2s}}, \quad \mathrm{Re}(s) > 1.$$

This is the Eisenstein series for $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. It has the following properties:

- $E(\gamma z, s) = E(z,s)$, $\gamma \in \mathrm{SL}(2,\mathbb{Z})$.
- $E(z,s)$ admits a meromorphic extension to $s \in \mathbb{C}$,
- $E(z,s)$ is holomorphic on $\mathrm{Re}(s) = 1/2$,
- $\Delta E(z,s) = s(1-s)E(z,s)$.

It follows that $r \in \mathbb{R} \mapsto E(z, 1/2 + ir)$ is a generalized eigenfunction. The map

$$f \in C_c^\infty(\mathbb{R}^+) \mapsto \frac{1}{2\pi}\int_0^\infty f(r)E(z,1/2+ir)\,dr$$

extends to an isometry $E\colon L^2(\mathbb{R}^+) \to L^2_{ac}(\Gamma\backslash\mathbb{H}^2)$, and

$$E^*(\varphi)(r) = \int_X \varphi(z)E(z,1/2-ir)\,d\mu(z), \quad \varphi \in C_c^\infty(\Gamma\backslash\mathbb{H}^2),$$

$$E^*(\Delta\varphi)(r) = (1/4+r^2)E^*(\varphi)(r), \quad \varphi \in \mathrm{dom}(\Delta).$$

Scattering matrix

Fourier expansion of $E(z,s)$:

$$E(x+iy,s) = y^s + C(s)y^{1-s} + O\left(e^{-cy}\right)$$

as $y \to \infty$. Sommerfeld radiation condition

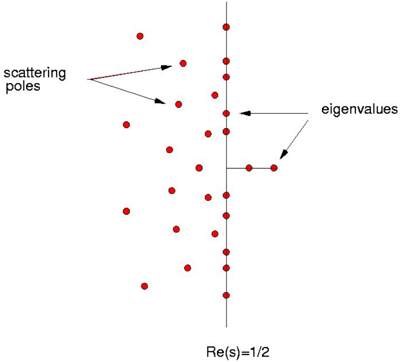
- $y^{1/2+ir}$ incoming plane wave, $y^{1/2-ir}$ outgoing plane wave, $E(z,1/2+ir)$ the distorted plane wave.
- $S(r) = C(1/2+ir)$ scattering matrix,
- $C(s)$ analytic continuation of the scattering matrix,

General surface: $a_k \mapsto E_k(z,s)$, $k = 1, \ldots, m$, Eisenstein series.
Scattering matrix: $C(s) = (C_{kl}(s))_{k,l=1}^m$.

Let

$$R_X(s) = (\Delta - s(1-s))^{-1}$$

Definition: The poles of $R_X(s)$ are called Resonances. Scattering resonances := poles of $C(s)$ = poles of $R_X(s)$ with $\mathrm{Re}(s) < 1/2$.



The poles are distributed in a strip of the form $-c < \mathrm{Re}(s) \leq 1$.

Distribution of resonances

Put

$$N_\Gamma(\lambda) = \#\{j \colon \lambda_j \leq \lambda^2\}, \quad \phi(s) := \det C(s).$$

Theorem 3 (Selberg): As $\lambda \to \infty$, we have

$$N_\Gamma(\lambda) - \frac{1}{4\pi}\int_{-\lambda}^{\lambda} \frac{\phi'}{\phi}\left(\frac{1}{2} + ir\right)\, dr \sim \frac{\mathrm{Area}(X)}{4\pi}\lambda^2.$$

proof: Selberg trace formula applied to the heat operator gives

$$\sum_j e^{-t\lambda_j} - \frac{1}{4\pi}\int_{\mathbb{R}} e^{-(1/4+r^2)t}\frac{\phi'}{\phi}\left(\frac{1}{2} + ir\right)\, dr \sim \frac{\mathrm{Area}(X)}{4\pi}t^{-1}$$

as $t \to 0+$. Furthermore, for $\lambda \gg 0$, the winding number

$$-\frac{1}{4\pi}\int_{-\lambda}^{\lambda} \frac{\phi'}{\phi}\left(\frac{1}{2} + ir\right)\, dr$$

is monotonic increasing. Tauberian theroem implies Theorem 3.

Distribution of resonances

Let $N_{scre}(\lambda)$ the number of scattering resonances in the circle of radius λ, counted with multiplicities.

Theorem 4 (Selberg): As $\lambda \to \infty$,

$$-\frac{1}{2\pi}\int_{-\lambda}^{\lambda} \frac{\phi'}{\phi}\left(\frac{1}{2}+ir\right)\, dr = N_{scre}(\lambda) + O(\lambda).$$

Note that

$$N_{res}(\lambda) := 2N_{\Gamma}(\lambda) + N_{scre}(\lambda)$$

is the counting function of resonances. By the above it satisfies Weyl's law

$$N_{res}(\lambda) \sim \frac{\mathrm{Area}(X_{\Gamma})}{2\pi}\lambda^2, \quad \lambda \to \infty.$$

Use of the full power of the trace formula gives an expansion

$$N_{res}(\lambda) = \frac{\mathrm{Area}(X_\Gamma)}{2\pi}\lambda^2 + c\lambda\log\lambda + O(\lambda), \quad \lambda \to \infty.$$

This is one of the rare cases where the counting function of the resonances has an asymptotic expansion.

In general, we know very little about the analytic properties of the scattering matrix. For the principal congruence subgroup $\Gamma(N)$, however, the entries of the scattering matrix can be expressed in terms of known functions of analytic number theory.

Huxley: For $\Gamma(N)$ we have

$$\det C(s) = (-1)^l A^{1-2s}\left(\frac{\boldsymbol{\Gamma}(1-s)}{\boldsymbol{\Gamma}(s)}\right)^k \prod_\chi \frac{L(2-2s,\bar{\chi})}{L(2s,\chi)},$$

where $k, l \in \mathbb{Z}$, $A > 0$, χ Dirichlet character mod k, $k|N$, $L(s,\chi)$ Dirichlet L-function with character χ.

Arithmetic groups

Especially, for $N = 1$ we have

$$C(s) = \sqrt{\pi}\frac{\Gamma(s - 1/2)\zeta(2s - 1)}{\Gamma(s)\zeta(2s)},$$

where $\zeta(s)$ denotes the Riemann zeta function.

Thus for $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ we get

$$\left\{\text{scattering resonances}\right\} = \left\{\frac{1}{2}\rho \colon \zeta(\rho) = 0,\ 0 < \mathrm{Re}(\rho) < 1\right\}.$$

A similar result holds for $\Gamma(N)$. By standard facts of analytic number theory, we get

$$N_{scre}(\lambda) = O(\lambda \log \lambda).$$

Arithmetic groups

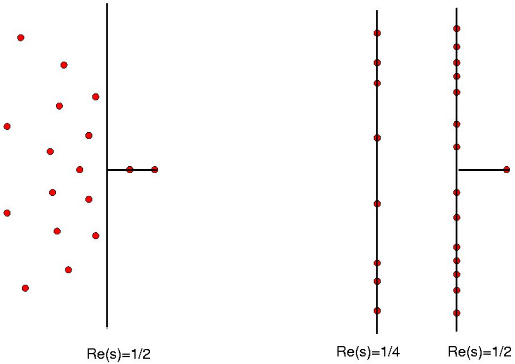
Theorem 5 (Selberg, 1956):

$$N_{\Gamma(N)}(\lambda) = \frac{\text{Area}(X(N))}{4\pi}\lambda^2 + O(\lambda \log \lambda), \quad \lambda \to \infty.$$

Thus for $\Gamma(N)$, L^2-eigenfunctions of Δ with eigenvalue $\lambda \geq 1/4$ ($=$ Maass automorphic cusp forms) exist in abundance.

Conjecture (Phillips, Sarnak, 1986): Except for the Teichmüller space of the once punctured torus, a generic Γ has only a finite number of eigenvalues.

- Thus for generic Γ the scattering resonances are expected to dominate in the counting function.
- What are the special properties of $\Gamma(N)$ that imply the existence of embedded eigenvalues ?
  The existence of Hecke operators



The left figure shows the expected distribution of resonances for a generic surfaces. The figure on the right shows the distribution of resonances for the modular surface $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^2$, under the assumption of the Riemann hypothesis. Except for the pole at $s = 1$, the scattering resonances are on the line $\mathrm{Re}(s) = 1/4$ and the poles corresponding to eigenvalues are on the line $\mathrm{Re}(s) = 1/2$.

Higher rank

ii) Higher rank
We consider now $X = \Gamma\backslash S = \Gamma\backslash G/K$ with $\mathrm{Rank}(S) > 1$.

Examples: 1) $X = \Gamma_1\backslash\mathbb{H}^2 \times \cdots \times \Gamma_m\backslash\mathbb{H}^2$, $m > 1$.

2) $X = \Gamma\backslash(\mathbb{H}^2 \times \cdots \times \mathbb{H}^2)$, $\Gamma \subset \mathrm{SL}(2,\mathbb{R})^n$ irreducible, Hilbert modular group, X has $\mathbb{Q}$-rank 1, intermediate case.

3) $G = \mathrm{SL}(n,\mathbb{R})$, $S_n = \mathrm{SL}(n,\mathbb{R})/SO(n)$, $n > 1$, $X = \Gamma(N)\backslash S_n$, where

$$\Gamma(N) = \{\gamma \in \mathrm{SL}(n,\mathbb{Z})\colon \gamma \equiv \mathrm{Id} \mod N\}.$$

Margulis: If $\mathrm{Rank}(G) > 1$, then every irreducible lattice in G is arithmetic (with an appropriate defintion of arithmetic).

Serre: Let $n > 2$. Then any subgroup $\Gamma \subset \mathrm{SL}(n,\mathbb{Z})$ of finite index is a congruence subgroup, i.e., there exists N such that $\Gamma(N) \subset \Gamma$.

The spectrum is now multidimensional. Let $G = NAK$ be the Iwasawa decompsotion, $\mathfrak{a} = \mathrm{Lie}(A)$, W the Weyl group. Then

$$\mathcal{D}(S) \cong S(\mathfrak{a}_{\mathbb{C}}^*)^W.$$

Therefore

$$\mathrm{spec}(\mathcal{D}(S)) \subset \mathfrak{a}_{\mathbb{C}}^*/W,$$

Eisenstein series are associated to rational parabolic subgroups $P \subset \mathrm{SL}(n, \mathbb{R})$.
A standard parabolic subgroup P of $\mathrm{SL}(n, \mathbb{R})$ is a stabilizer of a flag

$$\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_m \subset \mathbb{R}^n.$$

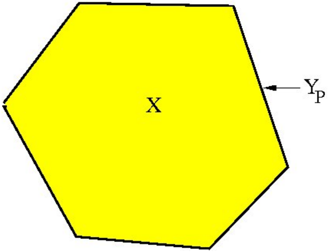
Geometric interpretation

Can be described by the Borel-Serre compactification

$$\overline{X}^{BS} = X \sqcup_{\{P\}} Y_P.$$

which is obtained by adding boundary faces Y_P at infinity.

- $\overline{X}^{BS}$ is a compact manifold with corners.



Each Y_P is a fibration

$$\pi_P \colon Y_P \to \Gamma_P \backslash S_P$$

over a locally symmetric space $\Gamma_P \backslash S_P$ of lower dimension with general fibre a compact nilmanifold $(\Gamma \cap N_P)\backslash N_P$.

- For $X = \Gamma\backslash\mathbb{H}^2$, $\overline{X}^{BS}$ is a compact surface with boundary, $\partial(\overline{X}^{BS}) = \sqcup_{i=1}^m S^1$.
- The boundary $\partial(\overline{X}^{BS})$ "parametrizes" the continuous spectrum.

Let $\phi \in L^2(\Gamma_P\backslash X_P)$ be an automorphic form, $\Lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$, $\mathrm{Re}(\Lambda) \gg 0$, there is an associated Eisenstein series $E(P, \phi, \Lambda)$

- $E(P, \phi, \Lambda)$ has a meromorphic extension to $\mathfrak{a}^*_{P,\mathbb{C}}$
- The asymptotic behavior of $E(P, \phi, \Lambda)$ near the boundary components $Y_{P'}$ determines the scattering matrices.
- multichannel scattering problem similar to N-body problem.

Langlands-Shahidi: For congruence groups Γ, the entries of the scattering matrices can be expressed in terms of automorphic L-functions.

Example: $X = \mathrm{SL}(3,\mathbb{Z})\backslash SL(3,\mathbb{R})/SO(3)$. $\partial(\overline{X}^{BS})$ has two components of maximal dimension Y_{P_1} and Y_{P_2}, which are torus fibrations

$$\pi_i\colon Y_{P_i} \to \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^2.$$

The associated scattering matrix $c_{P_2|P_1}(s)$, $s \in \mathbb{C}$, acts in the space of automorphic cusp forms on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^2$. Let $\phi \in L^2(\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^2)$ be a non-constant even eigenfunction of Δ. Then

$$\phi(x+iy) = \sum_{n=1}^{\infty} a_n y^{1/2} K_{ir}(2\pi ny)\cos(2\pi nx),$$

Let

$$L(s,\phi) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad \mathrm{Re}(s) > 1,$$

be the L-function attached to ϕ and $\Lambda(s,\phi)$ the completed L-function. Then

$$c_{P_2|P_1}(s)\phi = \frac{\Lambda(s,\phi)}{\Lambda(s+1,\phi)}\phi.$$

Weyl's law

Corollary: The automorphic L-functions occurring in the constant terms of Eisenstein series have meromorphic extensions to $\mathbb{C}$ and are of finite order.

Theorem 9 (Mü., 2007): For congruence subgroups of $\mathrm{SL}(n,\mathbb{Z})$, $n \geq 2$, the Eisenstein series and scattering matrices are meromorphic functions of order 1.
Let $S_n = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$, $n \geq 2$, and $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ a lattice. Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues of Δ_Γ in $L^2(\Gamma\backslash S_n)$.

$$N_\Gamma(\lambda) = \#\{j \colon \lambda_j \leq \lambda^2\}.$$

Thoerem 10 (Lapid, Mü., 2007): Let $d = \dim S_n$, $N \geq 3$. Then

$$N_{\Gamma(N)}(\lambda) = \frac{\mathrm{Vol}(\Gamma(N)\backslash S_n)}{(4\pi)^{d/2}\Gamma(d/2+1)}\lambda^d + O(\lambda^{d-1}(\log\lambda)^{\max(n,3)})$$

as $\lambda \to \infty$.

Method: Combination of Arthur's trace formula, which replaces the Selberg trace formula in the higher rank case, and Hörmanders method to estimate the spectral function of an elliptic operator.

Tempered spectrum and Ramanujan-Selberg conjecture

Let $\phi \in L^2(\Gamma\backslash S)$, $D\phi = \chi(D)\phi$, $D \in \mathcal{D}(S)$, where

$$\chi\colon \mathcal{D}(S) = S(\mathfrak{a}_{\mathbb{C}}^*)^W \to \mathbb{C}$$

is a character. Then

$$\chi \leftrightarrow \lambda \in \mathfrak{a}_{\mathbb{C}}^*/W.$$

- ϕ tempered $\Leftrightarrow \lambda \in i\mathfrak{a}^*/W$.

An automorphic form ϕ is called cusp form, if ϕ is rapidly decreasing.

Generalized Ramanujan-Selberg conjecture:
Every cusp form for $\mathrm{GL}(n)$ is tempered,

Let $\Lambda_{\mathrm{cus}}(\Gamma) \subset \mathfrak{a}_{\mathbb{C}}^*/W$ be the cuspidal spectrum.

Let $S_n = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ and $d_n = \dim S_n$.

Theorem 11 Lapid, Mü, 2007): Let $\Omega \subset i\mathfrak{a}^*$ be a bounded open subset with piecewise C^2 boundary. Let $\beta(\lambda)\, d\lambda$ be the Plancherel measure. Then

$$\sum_{\lambda \in \Lambda_{\mathrm{cus}}(\Gamma(N)), \lambda \in t\Omega} m(\lambda) = \frac{\mathrm{vol}(\Gamma(N)\backslash S_n)}{|W|} \int_{t\Omega} \beta(\lambda)\, d\lambda + O\left(t^{d_n - 1} (\log t)^{\max(n,3)}\right)$$

and

$$\sum_{\substack{\lambda \in \Lambda_{\mathrm{cus}}(\Gamma(N)) \\ \lambda \in B_t(0) \backslash i\mathfrak{a}^*}} m(\lambda) = O\left(t^{d_n - 2}\right)$$

as $t \to \infty$.

Problems:

1) Develope common framework to deal with scattering theory on locally symmetric spaces, N-body problem, and manifolds with corners.
2) Extend Theorem 10 to other groups. This depends on the analytic properties of the automorphic L-functions occurring in the constant terms of Eisenstein series.