The Arthur trace formula and spectral theory on locally symmetric spaces

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Introduction

The Selberg trace formula establishes a close relation between spectral and geometric data for finite volume locally symmetric spaces of rank 1.

For a general reductive group G over a number field F, Arthur, driven by Langlands' functoriallity conjectures, developed a trace formula for adelic quotients $G(F)\backslash G(\mathbb{A})$.

The key issue in Arthur's work is the comparison of the trace formulas of two different groups. However, it can also be used to study spectral problems on a single space. Such applications lead to new analytic problems related to the trace formula itself.

1. The Selberg trace formula

- G semisimple real Lie group with finite center of non-compact type
- $K \subset G$ maximal compact subgroup
- $\Gamma \subset G$ lattice
- R_Γ right regular representation of G in $L^2(\Gamma\backslash G)$, defined by

$$(R_\Gamma(g)f)(g') = f(g'g), \quad f \in L^2(\Gamma\backslash G).$$

Main goal: Study of the spectral resolution of $(R_\Gamma, L^2(\Gamma\backslash G))$.

a) Γ uniform lattice

Gelfand, Graev, Piateski-Shapiro: R_Γ decomposes discretely

$$R_\Gamma = \bigoplus_{\pi\in\widehat{G}} m_\Gamma(\pi)\pi.$$

Let $f \in C_c^\infty(G)$. Define

$$R_\Gamma(f) = \int_G f(g) R_\Gamma(g)\, dg.$$

Then $R_\Gamma(f)$ is an integral operator

$$\left(R_\Gamma(f)\varphi\right)(g) = \int_{\Gamma\backslash G} K_f(g, g')\varphi(g')\, dg', \quad \varphi \in L^2(\Gamma\backslash G),$$

with kernel

$$K_f(g, g') = \sum_{\gamma\in\Gamma} f(g^{-1}\gamma g').$$

Since $\Gamma\backslash G$ is compact, $R_\Gamma(f)$ is a trace class operator and

$$\operatorname{Tr} R_\Gamma(f) = \int_{\Gamma\backslash G} K_f(g, g)\, dg = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(g^{-1}\gamma g)\, dg.$$

- break the sum over γ into conjugacy classes $\{\gamma\}$ of Γ.

Let Γ_γ and G_γ be the centralizer of γ in Γ and G, respectively. The contribution of a conjugacy class $\{\gamma\}$ is

$$\int_{\Gamma_\gamma \backslash G} f(g^{-1}\gamma g)\, d\dot{g} = \mathrm{vol}(\Gamma_\gamma \backslash G_\gamma) I(\gamma, f),$$

where $I(\gamma, f)$ is the orbital integral

$$I(\gamma, f) = \int_{G_\gamma \backslash G} f(g^{-1}\gamma g)\, d\dot{g}, \quad f \in C_c^\infty(G).$$

Thus we get

$$\mathrm{Tr}\, R_\Gamma(f) = \sum_{\{\gamma\}} \mathrm{vol}(\Gamma_\gamma \backslash G_\gamma) I(\gamma, f).$$

On the other hand, by the result of Gelfand, Graev, and Piatetski-Shapiro, we get

$$\operatorname{Tr} R_\Gamma(f) = \sum_{\pi \in \widehat{G}} m_\Gamma(\pi) \operatorname{Tr} \pi(f).$$

Comparing the two expressions, we obtain

Trace formula (1. version):

$$\sum_{\pi \in \widehat{G}} m_\Gamma(\pi) \operatorname{Tr} \pi(f) = \sum_{\{\gamma\}} \operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) I(\gamma, f).$$

spectral side = geometric side

- $I(\gamma, f)$ and $\operatorname{Tr} \pi(f)$ are invariant distributions on G, i.e., invarinat under $f \to f^g$, where $f^g(g') = f(g^{-1} g' g)$.
- Fourier inversion formula can be used to express $I(\gamma, f)$ in terms of characters.

The rank one case.

To make the trace formula useful, one has to understand the distributions $I(\gamma, f)$ and $\operatorname{Tr}\pi(f)$ and to express them in differential geometric terms. This is possible if the $\mathbb{R}$-rank of G is 1.

We specialize to: $G = \mathrm{SL}(2, \mathbb{R})$, $K = \mathrm{SO}(2)$.

- $\mathbb{H} = G/K$ upper half-plane, $\Gamma \subset G$ co-compact.

Let

$$f \in C_c^\infty(G//K) = \{f \in C_c^\infty(G) \colon f(k_1 g k_2) = f(g),\ k_1, k_2 \in K\}.$$

Then $\operatorname{Tr}\pi(f) = 0$, unless π has a K-fixed vextor. Hence

$$\operatorname{Tr}\pi(f) \neq 0 \Leftrightarrow \exists\, s \in i\mathbb{R} \cap [-1, 1] \colon \pi = \pi_s.$$

Let

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right), \quad x + iy \in \mathbb{H}.$$

- hyperbolic Laplaceoperator on $\mathbb{H}$.

- Δ has discrete spectrum in $L^2(\Gamma\backslash\mathbb{H})$.

$$\sigma(\Delta)\colon 0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \to \infty.$$

- $m(\lambda_j)$ multiplicity of λ_j.

Frobenius reciprocity: $m(\pi_s) = m((1-s^2)/4)$ for $s \in i\mathbb{R} \cup [-1,1]$.

Let

$$\mathcal{A}(f)(t) = \int_{\mathbb{R}} f\left(\begin{pmatrix} e^{t/2} & x \\ 0 & e^{-t/2} \end{pmatrix}\right) dx$$

be the Abel transform of f. Then $\mathcal{A}$ defines an isomorphism

$$\mathcal{A}\colon C_c^\infty(G//K) \to C_c^\infty(\mathbb{R})^{even}.$$

Moreover $\mathcal{A}(f)$ is closely related to the orbital interal of f:

$$I(a_t, f) = \frac{1}{|e^t - e^{-t}|}\mathcal{A}(f)(2t), \quad a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

Let $h = \mathcal{A}(f)$. Then

$$\widehat{h}(r) = \int_G f(g)\phi_{1/2+ir}(g)\,dg,$$

is the spherical Fourier transform of f, where ϕ_λ is the spherical function.
f can be recovered from h by Plancherel inversion:

$$f(e) = \int_{\mathbb{R}} \widehat{h}(r) r \tanh(r)\,dr.$$

Moreover, using the polar decomposition $G = KAK$, it follows that

$$\widehat{h}(r) = \operatorname{Tr} \pi_{2ir}(f).$$

Assumption: Γ torsion free

- $\gamma \in \Gamma - \{e\}$ is hyperbolic,
- $\{\gamma\}$ corresponds to unique closed geodesic τ_γ in $\Gamma\backslash\mathbb{H}$.
- $\ell(\gamma) = \mathrm{length}(\tau_\gamma)$.

Let $\gamma \sim \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$. Then $\ell(\gamma) = t$.
Write the eigenvalues of Δ as

$$\lambda_j = \frac{1}{4} + r_j^2, \quad r_j \in \mathbb{R} \cap i[-1/2, 1/2].$$

Each γ can be uniquely written as $\gamma = \gamma_0^k$, $k \in \mathbb{N}$, where γ_0 is primitive. Then

$$\mathrm{vol}(\Gamma_\gamma \backslash G_\gamma) = \ell(\gamma_0).$$

Selberg's trace formula (K-invariant form):

$$\sum_j m(\lambda_j)\widehat{h}(r_j) = \frac{\text{Area}(\Gamma\backslash\mathbb{H})}{2\pi}\int_{\mathbb{R}} \widehat{h}(r) r \tanh(\pi r)\, dr + \sum_{\{\gamma\}\neq e} \frac{\ell(\gamma_0)}{e^{\ell(\gamma)/2} - e^{-\ell(\gamma)/2}} h(\ell(\gamma)).$$

- The kernel function $f \in C_c^\infty(G//K)$ has been eliminated from the formula.
- $h \in C_c^\infty(\mathbb{R})$.

b) Γ non-uniform

We assume that $\mathrm{vol}(\Gamma\backslash G) < \infty$ and $\Gamma\backslash G$ non-compact.

- $R_\Gamma(f)$ is not trace class
- R_Γ does not decompose discretely.

Langlands's theory of Eisenstein series provides a decomposition into invariant subspaces

$$L^2(\Gamma\backslash G) = L^2_d(\Gamma\backslash G) \oplus L^2_{ac}(\Gamma\backslash G),$$

where

$$R^d_\Gamma = \bigoplus_{\pi\in\widehat{G}} m_\Gamma(\pi)\pi,$$

and $L^2_d(\Gamma\backslash G)$ is the maximal invariant subspace, in which R_Γ decomposes discretely.

- $L^2_{ac}(\Gamma\backslash G)$ is described in terms of Eisenstein series.

Theorem. (Ji, Mü, '98): *For each $f \in C_c^\infty(G)$, $R_\Gamma^d(f)$ is a trace class operator.*

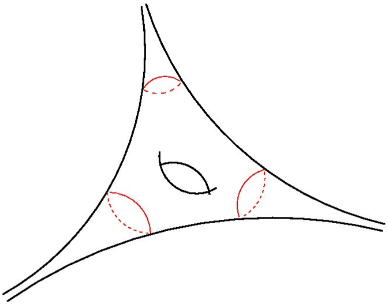
Therefore

$$\operatorname{Tr} R_\Gamma^d(f) = \sum_{\pi \in \widehat{G}} m_\Gamma(\pi) \operatorname{Tr} \pi(f).$$

- In higher rank, there is no trace formula within this framework.

The rank one case: $G = \mathrm{SL}(2, \mathbb{R})$, $\Gamma \subset G$ a non-uniform lattice.

- Δ has continuous spectrum: $[1/4, \infty)$,
- possible eigenvalues of Δ: $0 = \lambda_0 < \lambda_1 < \cdots$,
- the only obvious eigenfunction is the constant function for which $\lambda = 0$.
- continuous spectrum is described by Eisenstein series.



A hyperbolic surface with 3 cusps.

The surface X can be compactified by adding m points $a_1, \ldots, a_m$:

$$\overline{X} = X \cup \{a_1, \ldots, a_m\}.$$

- $\overline{X}$ is a closed Riemann surface.
- The points $a_1, \ldots, a_m$ are called cusps. They correspond to parabolic fixed points $p_1, \ldots, p_m \in \mathbb{R} \cup \{\infty\}$ of Γ.
- $a_k \mapsto E_k(z, s)$, Eisenstein series attached to a_k.

Example: $\Gamma = \mathrm{SL}(2, \mathbb{Z})$.

- $\Gamma\backslash\mathbb{H}$ has a single cusp ∞.
- Eisenstein series attached to ∞:

$$E(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma z)^s = \sum_{(m,n)=1} \frac{y^s}{|mz + n|^{2s}}, \quad \mathrm{Re}(s) > 1.$$

Properties:

- $E(\gamma z, s) = E(z, s)$, $\gamma \in \mathrm{SL}(2, \mathbb{Z})$.
- $E(z, s)$ admits a meromorphic extension to $s \in \mathbb{C}$,
- $E(z, s)$ is holomorphic on $\mathrm{Re}(s) = 1/2$,
- $\Delta E(z, s) = s(1 - s)E(z, s)$.

It follows that $r \in \mathbb{R} \mapsto E(z, 1/2 + ir)$ is a generalized eigenfunction.

Scattering matrix

Fourier expansion of $E(z,s)$:

$$E(x+iy,s) = y^s + C(s)y^{1-s} + O\left(e^{-cy}\right)$$

as $y \to \infty$. Sommerfeld radiation condition

- $y^{1/2+ir}$ incoming plane wave, $y^{1/2-ir}$ outgoing plane wave, $E(z,1/2+ir)$ the distorted plane wave.
- $S(r) = C(1/2+ir)$ scattering matrix,
- $C(s)$ analytic continuation of the scattering matrix,

General case: $E_k(z,s)$, $k = 1, ..., m$, Eisenstein series. Fourier expansion of $E_k(z,s)$ in the cusp a_l gives scattering matrix:

$$C(s) = (C_{kl}(s))_{k,l=1}^{m}.$$

Let $\phi(s) = \det C(s)$.

$$\frac{1}{4\pi}\int_{\mathbb{R}} \widehat{h}(r)\frac{\phi'}{\phi}(1/2+ir)\,dr$$

contribution of the Eisenstein series to the trace formula.

- Γ has now parabolic elements

Parabolic contribution:

$$\int_{\mathbb{R}} \widehat{h}(r)\frac{\Gamma'}{\Gamma}(1+ir)\,dr.$$

Selberg trace formula for non-uniform lattices:

$$\sum_j \widehat{h}(r_j) - \frac{1}{4\pi}\int_{-\infty}^{\infty} \widehat{h}(r)\frac{\phi'}{\phi}(1/2+ir)\,dr + \frac{1}{4}\phi(1/2)h(0)$$

$$= \frac{\mathrm{Area}(\Gamma\backslash\mathbb{H})}{4\pi}\int_{\mathbb{R}} \widehat{h}(r) r\tanh(\pi r)\,dr + \sum_{\{\gamma\}\neq e} \frac{\ell(\gamma_0)}{e^{\ell(\gamma)/2} - e^{-\ell(\gamma)/2}} h(\ell(\gamma))$$

$$- \frac{m}{2\pi}\int_{-\infty}^{\infty} \widehat{h}(r)\frac{\Gamma'}{\Gamma}(1+ir)dr + \frac{m}{4}\widehat{h}(0) - m\ln 2\; h(0).$$

- Can be understood as relative trace formula

II. Applications of the trace formula

1) Weyl's law and the existence of cups forms

Rank one case: $G = \mathrm{SL}(2,\mathbb{R})$, $\Gamma \subset G$ non-uniform lattice.
Let $0 = \lambda_0 < \lambda_1 \leq \cdots$ be the eigenvalues of Δ, $C(s)$ scattering matrix, $\phi(s) = \det C(s)$. Put

$$N_\Gamma(\lambda) = \#\{j \colon \lambda_j \leq \lambda^2\}, \quad M_\Gamma(\lambda) = -\frac{1}{4\pi}\int_{-\lambda}^{\lambda} \frac{\phi'}{\phi}(1/2 + ir)\, dr.$$

Theorem 1 (Selberg): As $\lambda \to \infty$, we have

$$N_\Gamma(\lambda) + M_\Gamma(\lambda) = \frac{\mathrm{Area}(\Gamma\backslash\mathbb{H})}{4\pi}\lambda^2 + O(\lambda \log \lambda).$$

proof: (without remainder term)

- k_t kernel of the heat operator $e^{-t\widetilde{\Delta}}$ on $\mathbb{H}$.
- $k_t \in \mathcal{C}^1(G//K)$ (bi-K-invariant, integrable, rapidely decreasing functions).
- Selberg trace formula can be applied to k_t.

Let $h_t = \mathcal{A}(k_t)$ be the Abel transform. Then

$$h_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-t/4 - x^2/(4t)}, \quad \widehat{h}_t(r) = e^{-(1/4+r^2)t}.$$

If we insert h_t in the trace formula, we get

$$\sum_j e^{-t\lambda_j} - \frac{1}{4\pi} \int_{\mathbb{R}} e^{-(1/4+r^2)t} \frac{\phi'}{\phi}\left(\frac{1}{2} + ir\right) \, dr \sim \frac{\text{Area}(X)}{4\pi} t^{-1}$$

as $t \to 0+$.

- For $\lambda \gg 0$, the winding number $M_\Gamma(\lambda)$ is monotonic increasing.
- Tauberian theroem $\Rightarrow$ Theorem.

A more sophisticated use of the trace formula gives an estimation of the remainder term.
First step is to estimate the number of eigenvalues in an interval.
Hörmander's method.

The scattering matrix for arithmetic groups

- In general, $N_\Gamma(\lambda)$ and $M_\Gamma(\lambda)$ can not be separated.
- For the principal congruence subgroup $\Gamma(N)$, the entries of the scattering matrix can be expressed in terms of known functions of analytic number theory.

Huxley: For $\Gamma(N)$ we have

$$\phi(s) = (-1)^l A^{1-2s} \left(\frac{\mathbf{\Gamma}(1-s)}{\mathbf{\Gamma}(s)} \right)^k \prod_{\chi} \frac{L(2-2s, \bar{\chi})}{L(2s, \chi)},$$

where $k, l \in \mathbb{Z}$, $A > 0$, χ Dirichlet character mod k, $k|N$, $L(s, \chi)$ Dirichlet L-function with character χ.
Especially, for $N = 1$ we have

$$\phi(s) = \sqrt{\pi} \frac{\mathbf{\Gamma}(s - 1/2)\zeta(2s-1)}{\mathbf{\Gamma}(s)\zeta(2s)},$$

where $\zeta(s)$ denotes the Riemann zeta function.

Thus for $\Gamma(N)$ we get

$$\left|\frac{\phi'}{\phi}(1/2+ir)\right| \ll \log^k(|r|+1), \quad r \in \mathbb{R},$$

and therefore

$$M_{\Gamma(N)}(\lambda) = O(\lambda \log \lambda).$$

Theorem 2 (Selberg, 1956):

$$N_{\Gamma(N)}(\lambda) = \frac{\operatorname{Area}(\Gamma(N)\backslash\mathbb{H})}{4\pi}\lambda^2 + O(\lambda\log\lambda), \quad \lambda \to \infty.$$

- For $\Gamma(N)$, L^2-eigenfunctions of Δ with eigenvalue $\lambda \geq 1/4$ ($=$ Maass automorphic cusp forms) exist in abundance.
- For $\Gamma(1) = \mathrm{SL}(2,\mathbb{Z})$ no eigenfunction with eigenvalue $\lambda > 0$ can be constructed explicitly.

2) Distribution of Hecke eigenvalues

$S_k(\Gamma(1))$ space of cusp forms of weight k.

$$T_n \colon S_k(\Gamma(1)) \to S_k(\Gamma(1))$$

the n-th Hecke operator.

- $\mathcal{S}_k$ the set of all normalized Hecke eigenforms $f \in S_k(\Gamma(1))$.

Then

$$T_n f = a_f(n) f, \quad f \in \mathcal{S}_k.$$

Put $\lambda_f(n) = n^{(1-k)/2} a_f(n)$.
Deligne: $\lambda_f(p) \in [-2, 2]$ for p prime.

Conjecture (Serre): For each $h \in C([-2, 2])$

$$\frac{1}{\pi(x)} \sum_{p \le x} h(\lambda_f(p)) \to \frac{1}{2\pi} \int_{-2}^{2} h(t) \sqrt{4 - t^2}\, dt, \quad x \to \infty.$$

Sato-Tate conjecture for modular forms.

Theorem (H. Nagoshi, 2006): Suppose that $k = k(x)$ satisfies $\frac{\log k}{\log x} \to \infty$ as $x \to \infty$. Then for every $h \in C([-2,2])$, we have

$$\frac{1}{\pi(x)\#S_k} \sum_{\substack{p \leq x \\ f \in S_k}} h(\lambda_f(p)) \to \frac{1}{2\pi} \int_{-2}^{2} h(t)\sqrt{4 - t^2}\, dt, \quad x \to \infty.$$

4) Limit multiplicities

a) $\Gamma \subset G$ uniform lattice

- $\Gamma = \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_n \supset \cdots$ tower of normal subgroups of finte index, $\cap_j \Gamma_j = \{e\}$.

$$R_{\Gamma_j} = \widehat{\bigoplus}_{\pi \in \widehat{G}} m(\Gamma_j, \pi)\pi.$$

- $S \subset \widehat{G}$ open, relative compact, regular for the Plancherel measure μ_{PL}.

Put

$$\mu_j(S) = \frac{1}{\mathrm{vol}(\Gamma_j \backslash G)} \sum_{\pi \in S} m(\Gamma_j, \pi).$$

deGeorge-Wallach, Delorme: $\lim_{j \to \infty} \mu_j(S) = \mu_{PL}(S)$.
b) $\Gamma \subset G$ non-uniform lattice
Savin: $\pi \in \widehat{G}_d$.

$$\lim_{j \to \infty} \mu_j(\{\pi\}) = d(\pi).$$

Clozel: weak version.

$$\liminf_{j \to \infty} \mu_j(\{\pi\}) > \varepsilon > 0.$$

5) Low lying zeros of L-functions

$f \in S_k$, $L(s,f)$ L-function attached to f, ϕ test function

$$D(f,\phi) = \sum_{\gamma} \phi(\gamma),$$

where γ ranges over normalized zeros of $L(s,f)$. $\mathcal{F}(Q)$ family of L-functions depending on parameter Q.

$$E(\mathcal{F}(Q), f) = \frac{1}{|\mathcal{F}(Q)|} \sum_{f \in \mathcal{F}(Q)} D(f,\phi).$$

Behavior as $Q \to \infty$.

5) Jacquet-Langlands

Correspondence between automorphic forms of a quarternion algebra and GL(2).

$$(\Gamma \backslash \mathbb{H}) \leftrightarrow (\Gamma' \backslash \mathbb{H})$$
$$\{\lambda_j, t_{p,j}\} \leftrightarrow \{\lambda_j', t_{p,j}'\}$$

$\Gamma'\backslash\mathbb{H}$ is a compact Riemann surface attached to a congruence quaternion group Γ', $\Gamma\backslash H$ non-compact congruence surface.

III. Higher rank

- $S = G/K$, Δ Laplace operator on $\Gamma\backslash S$.
- $L^2_{\rm cus}(\Gamma\backslash S) \subset L^2(\Gamma\backslash S)$ closure of the span of the space of cusp forms.
- Δ has discrete spectrum in $L^2_{\rm cus}(\Gamma\backslash S)$.

$$L^2_{\rm dis}(\Gamma\backslash S) = L^2_{\rm cus}(\Gamma\backslash S) \oplus L^2_{\rm res}(\Gamma\backslash S).$$

$N_\Gamma^{\rm cus}(\lambda)$, $N_\Gamma^{\rm res}(\lambda)$ counting function of cuspidal and residual spectrum, resp.

General results:

Theorem (Donnelly, '82): Let $d = \dim S$.

$$\limsup_{\lambda\to\infty} \frac{N_\Gamma^{\rm cus}(\lambda)}{\lambda^{d/2}} \le \frac{\mathrm{vol}(\Gamma\backslash S)}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}+1\right)}.$$

Theorem (Mü, '89): $N_\Gamma^{\mathrm{res}}(\lambda) \ll \lambda^{2d}, \quad \lambda \geq 1.$

Conjecture 1 (Sarnak): $\operatorname{rank}(S) > 1$. Then $N_\Gamma^{\mathrm{cus}}(\lambda)$ satisfies Weyl's law.

Conjecture 2: $N_\Gamma^{\mathrm{res}}(\lambda) \ll \lambda^{(d-1)/2}$.

Theorem (Lindenstrauss,Venkatesh): $\mathbf{G}$ split adjoint semisimple group over $\mathbb{Q}$, $G = \mathbf{G}(\mathbb{R})$, $\Gamma \subset \mathbf{G}(\mathbb{Q})$ a congrunece group, $d = \dim S$. Then

$$N_\Gamma^{\mathrm{cus}}(\lambda) \sim \frac{\operatorname{vol}(\Gamma\backslash S)}{(4\pi)^{d/2}\Gamma(d/2+1)}\lambda^{d/2}, \quad \lambda \to \infty.$$

- Confirms the conjectur of Sarnak in these cases.

Previous results:

S. Miller: $G = \mathrm{SL}(3,\mathbb{R})$, $\Gamma = \mathrm{SL}(3,\mathbb{Z})$,

Mü: $G = \mathrm{SL}(n,\mathbb{R})$, $\Gamma = \Gamma(N)$.

Estimation of the remainder term

Theorem (Lapid, Mü, 2007): Let $S_n = \mathrm{SL}(n,\mathbb{R})/\,\mathrm{SO}(n)$, $d = \dim S_n$, $N \geq 3$. Then

$$N^{\mathrm{cus}}_{\Gamma(N)}(\lambda) = \frac{\mathrm{vol}(\Gamma(N)\backslash S)}{(4\pi)^{d/2}\Gamma(d/2+1)}\lambda^{d/2} + O\left(\lambda^{(d-1)/2}(\log\lambda)^{\max(n,3)}\right).$$

Method: Combination of Hörmander's method and Arthur's trace formula.

Mœglin, Waldsburger, 1989: Description of the residual spectrum of $\mathrm{GL}(n)$.

Combined with Donnelly's estimate, we get

Theorem (Mœglin, Waldsburger, 1989): $S_n = \mathrm{SL}(n,\mathbb{R})/\,\mathrm{SO}(n)$, $d = \dim S_n$.

$$N^{\mathrm{res}}_{\Gamma(N)}(\lambda) \ll \lambda^{d/2-1}.$$

Multidimensional version

- $G = NAK$ Iwasawa decomposition, $\mathfrak{a} = \mathrm{Lie}(A)$, $H\colon G \to \mathfrak{a}$, $H(nak) = \log a$, $W = W(G, A)$.
- $\mathcal{D}(S)$ ring of invariant differential operators on S.

Harish-Chandra: $\mathcal{D}(S) \cong S(\mathfrak{a}_\mathbb{C})^W$.
Thus, if

$$\chi\colon \mathcal{D}(S) = S(\mathfrak{a}_\mathbb{C}^*)^W \to \mathbb{C}$$

is a character. Then

$$\chi = \chi_\lambda \leftrightarrow \lambda \in \mathfrak{a}_\mathbb{C}^*/W.$$

For $\lambda \in \mathfrak{a}_\mathbb{C}^*$ let

$$\mathcal{E}_{\mathrm{cus}}(\lambda) = \left\{\varphi \in L^2_{\mathrm{cus}}(\Gamma\backslash S)\colon D\varphi = \chi_\lambda(D)\varphi,\ D \in \mathcal{D}(S)\right\}$$

Lwt $m_{\mathrm{cus}}(\lambda) = \dim \mathcal{E}_{\mathrm{cus}}(\lambda)$. Then the cuspidal spectrum is defined as

$$\Lambda_{\mathrm{cus}}(\Gamma) = \{\lambda \in \mathfrak{a}_\mathbb{C}^*/W\colon m(\lambda) > 0\}.$$

- $\Lambda_{\mathrm{cus}}(\Gamma) \cap i\mathfrak{a}^*/W$ is the tempered spectrum
- $\Lambda_{\mathrm{cus}}(\Gamma) - (\Lambda_{\mathrm{cus}}(\Gamma) \cap i\mathfrak{a}^*/W)$ the complementary spectrum.

Theorem (Lapid, Mü, 2007): Let $S_n = \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ and $d_n = \dim S_n$, $\Omega \subset i\mathfrak{a}^*$ a bounded open subset with piecewise C^2 boundary, $\beta(\lambda)$ be the Plancherel measure. Then as $t \to \infty$

$$\sum_{\lambda \in \Lambda_{\mathrm{cus}}(\Gamma(N)), \lambda \in t\Omega} m(\lambda) = \frac{\mathrm{vol}(\Gamma(N) \backslash S_n)}{|W|} \int_{t\Omega} \beta(\lambda)\, d\lambda + O\left(t^{d_n - 1} (\log t)^{\max(n,3)}\right)$$

and

$$\sum_{\substack{\lambda \in \Lambda_{\mathrm{cus}}(\Gamma(N)) \\ \lambda \in B_t(0) \setminus i\mathfrak{a}^*}} m(\lambda) = O\left(t^{d_n - 2}\right).$$

Duistermaat, Kolk, Varadarajan, 1979: This results holds for G arbitrary, and $\Gamma \subset G$ a uniform lattice.

IV. Problems

1) Generalize the results of Duistermaat-Kolk-Varadarajan on spectral asymptotics for compact locally symmetric spaces $\Gamma\backslash S$ to non-compact quotients where Γ is a congruence subgroup. In particular, establish Weyl's law with a remainder term.

2) Analyze the spectral asymptotics for the Bochner-Laplace operator acting on the sections of a locally homogeneous vector bundle over $\Gamma\backslash S$ (i.e. automorphic forms with a given K_∞-type).

3) Study the distribution of Hecke eigenvalues.

4) Study the distribution of low-lying zeros of L-functions of Hecke eigenforms of $\mathrm{SL}(n,\mathbb{Z})\backslash \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ with large eigenvalues.

5) Study the limiting behavior of the Laplace spectrum for towers $\Gamma_1 \supset \Gamma_2 \supset \cdots$.

V. The Arthur trace formula

- The Arthur trace formula is the main tool to study these problems in the higher rank case.
- General reductive group needs adelic framwork.

G reductive algebraic group over $\mathbb{Q}$, $\mathbb{A} = \prod'_v \mathbb{Q}_v$ ring of adels of $\mathbb{Q}$, $G(\mathbb{A}) = \prod'_v G(\mathbb{Q}_v)$.
We study now the spectral resolution of the regular representation

$$R\colon G(\mathbb{A}) \to \mathrm{Aut}(L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))).$$

This is related to the previous framework as follows. Let $K_f \subset \prod_{p<\infty} G(\mathbb{Z}_p)$ be an open compact subgroup. Then

$$G(\mathbb{Q})\backslash G(\mathbb{A})/K_f \cong \bigsqcup_j (\Gamma_j\backslash G(\mathbb{R}))\,.$$

$$G(\mathbb{A})^1 = \bigcap_{\chi \in X(G)_{\mathbb{Q}}} \ker|\chi|, \quad G(\mathbb{A}) = G(\mathbb{A})^1 \cdot A_G(\mathbb{R})^0.$$

The (non-invariant) trace formula is an identity of distributions on $G(\mathbb{A})^1$

$$\sum_{\chi \in \mathfrak{X}} J_\chi(f) = \sum_{\mathfrak{o} \in \mathfrak{O}} J_{\mathfrak{o}}(f), \quad f \in C_0^\infty(G(\mathbb{A})^1)$$

spectral side = geometric side

- $\mathfrak{X}$ set of cuspidal data; equivalence classes of (M, ρ), M Levi factor of rational parabolic subgroup, ρ cuspidal automorphic representation of $M(\mathbb{A})^1$.
- $\mathfrak{O}$ set of equivalence classes in $G(\mathbb{Q})$, $\gamma \sim \gamma'$, if γ_s and γ'_s are $G(\mathbb{Q})$-conjugate.

Spectral side

- J_χ is derived from the constant terms of Eisenstein series and generalizes
$$\frac{1}{4\pi}\int_{\mathbb{R}} \widehat{h}(r)\frac{\phi'}{\phi}(1/2+ir)\,dr$$
- $P \subset G$ $\mathbb{Q}$-parabolic subgroup, $P = M_P N_P$ Levi decomposition
- $A_P \subset M_P$ split component of the center of M_P, $\mathfrak{a}_P = \mathrm{Lie}(A_P)$
- $\mathcal{A}^2(P)$ square integrable automorphic forms on $N_P(\mathbb{A})M_P(\mathbb{Q})\backslash G(\mathbb{A})$
- $Q = M_Q N_Q$ $\mathbb{Q}$-parabolic subgroup of G, $M_P = M_Q = M$.

$$M_{Q|P}(\lambda)\colon \mathcal{A}^2(P) \to \mathcal{A}^2(Q), \quad \lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$$

intertwining operator, meromorphic function of λ, main ingredient of J_χ.

- $\pi \in \Pi(M(\mathbb{A})^1)$ determines subspace $\mathcal{A}^2_\pi(P) \subset \mathcal{A}^2(P)$ of automorphic forms which transform according to π
- $M_{Q|P}(\lambda, \pi)$ restriction of $M_{Q|P}(\lambda)$ to $\mathcal{A}^2_\pi(P)$.
- $\rho_\pi(P, \lambda)$ induced representation of $G(\mathbb{A})$ in $\overline{\mathcal{A}}^2_\pi(P)$.

Let P be maximal parabolic and $\overline{P}$ the opposite parabolic group. Then the following integral-series is part of the spectral side

$$\sum_{\pi \in \Pi_{\mathrm{cus}}(M(\mathbb{A})^1)} \int_{-\infty}^{\infty} \operatorname{Tr}\left(M_{\overline{P}|P}(i\lambda, \pi)^{-1} \frac{d}{dz} M_{\overline{P}|P}(i\lambda, \pi) \rho_\pi(P, i\lambda, f) \right) d\lambda$$

Problem: Absolute convergence of the integral-series.

$\pi = \otimes_v \pi_v$, $\phi \in \mathcal{A}^2_\pi(P)$, $\phi = \otimes_v \phi_v$. S finite set of places, containing ∞, such that ϕ_v is fixed under $G(\mathbb{Z}_p)$ for $p \notin S$. There exist finite-dimensional representations $r_1, \dots, r_m$ of ${}^L M$ such that

$$M_{\overline{P}|P}(s,\pi)\phi = \bigotimes_{v \in S} M_{\overline{P}|P}(s,\pi_v)\phi_v \otimes \bigotimes_{v \notin S} \widetilde{\phi}_v \cdot \prod_{i=1}^m \frac{L_S(is,\pi,\widetilde{r}_i)}{L_S(1+is,\pi,\widetilde{r}_i)},$$

where

$$L_S(s,\pi,r) = \prod_{v \notin S} L(s,\pi_v,r_v), \quad \mathrm{Re}(s) \gg 0,$$

is the partial automorphic L-function attached to π and r.

- This reduces the problem to the estimation of the number of zeros of $L_S(s,\pi,\widetilde{r}_j)$ in a circle of radius T as $T \to \infty$.
- Need to control the constants in terms of π.

Lapid, Mü, 2008: In general, the study of the distribution J_χ can be reduced to the study of integrals as above associated to maximal parabolics in Levi subgroups.

Theorem (Lapid, Mü, 2008): For every reductive group G, the spectral side of the trace formula is absolutely convergent.

Mü, Speh, Lapid, 2004: $G = GL(n)$.

- This is a first step.
- The intended applications of the trace formula to spectral problems require a finer analysis of the L-functions.

For $\mathrm{GL}(n)$ the relevant L-functions are the Rankin-Selberg L-functions $L(s, \pi_1 \times \pi_2)$ attached to cuspidal automorphic representations π_i of $\mathrm{GL}(n_i, \mathbb{A})$, $i = 1, 2$, $n = n_1 + n_2$.

Jacquet, Shahidi, Mœglin/Waldspurger, ...: completed L-function $\Lambda(s, \pi_1 \times \pi_2)$ has at most simple poles at $s = 0, 1$, $s(1-s)\Lambda(s, \pi_1 \times \pi_2)$ is entire of order 1, satisfies functional equation.

Geometric side

The distributions $J_{\mathfrak{o}}$ are given in terms weighted orbital integrals. In general, they are difficult to define. A special case is

$$\int_{G_\gamma \backslash G} f(g^{-1}\gamma g) w(g)\, dg,$$

where $w(g)$ is a certain weight function.

- Weighted orbital integrals are non-invariant distributions.