— ENTWURF —

Klausur Topologie II SS 2017

Streng vertraulich !!!

Aufgabe 1. (Questionnaire) [1 point each]

Answer the following questions **without** arguments.

(1) $H^1(X;\mathbb{Z}) \cong \pi_1(X,x_0)^{\mathrm{ab}}$ for a path-connected space X. .... Yes ☐ No ☐

(2) $H_n(X;\mathbb{Z}) = 0$ for a CW complex of dimension $\dim(X) < n$. .... Yes ☐ No ☐

(3) $H^n(X \times Y;\mathbb{Q}) \cong H^n(X;\mathbb{Q}) \otimes H^n(Y;\mathbb{Q})$ for each n, X or Y of finite type. ...... Yes ☐ No ☐

(4) $(\alpha \cup \beta) \cap a = \alpha \cap (\beta \cap a)$ for $\alpha, \beta \in H^*(X)$ and $a \in H_*(X)$. .... Yes ☐ No ☐

(5) The functor $M \mapsto M \otimes_{\mathbb{Z}} \mathbb{G}$ is right exact (on the category of abelian groups). .. Yes ☐ No ☐

(6) $H_n(X;\mathbb{Z})$ is finitely generated for all n, if X is a compact CW complex. .... Yes ☐ No ☐

(7) $\chi(X \times Y) = \chi(X)\,\chi(Y)$ whenever defined. .... Yes ☐ No ☐

(8) $H_n(\tilde{X}) \cong (H_n(X))^l$ for an l-fold covering $\tilde{X} \to X$. .... Yes ☐ No ☐

(9) $\mathrm{Tor}(\mathbb{Z}/4, \mathbb{Z}/6) \cong \mathbb{Z}/2$. .... Yes ☐ No ☐

(10) Any submanifold of an orientable manifold is orientable. .... Yes ☐ No ☐

Aufgabe 2. (Simplicial approximation of maps) [5 + 5 points]

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two simplicial complexes and denote by $|\mathfrak{X}|$ resp. $|\mathfrak{Y}|$ their geometric realizations.

(1) Formulate the Simplicial Approximation Theorem for Continuous Maps $f\colon |\mathfrak{X}| \to |\mathfrak{Y}|$.

(2) Application: If $0 < k < n$, any map $f\colon \mathbb{S}^k \to \mathbb{S}^n$ is null-homotopic.

Aufgabe 3. (Lefschetz number) [2+2+4+2 points]

(1) Define the Lefschetz number $L(f)$ of a self-map $f\colon X \to X$ whenever it can be defined.

(2) Formulate the Lefschetz Fix Point Theorem.

(3) Prove this theorem (assuming, that f is already a simplicial map).

(4) Let f be a self-map of X and g a self-map of Y, both finite polyhedra. Show the product formula
$$L(f \times g) = L(f)\, L(g)\,.$$

Aufgabe 4. (Universal Coefficient Theorem in Homology) [1+3+2+4 points]

Let $\mathbb{K}$ be a principal ideal domain.

(1) How is the homology with coefficients in a $\mathbb{K}$-module $\mathbb{G}$ defined ?

(2) Formulate the Universal Coeffcient Theorem for free chain complexes over $\mathbb{K}$.

(3) Derive the Universal Coefficient Theorem for spaces with coefficients in a $\mathbb{K}$-module $\mathbb{G}$.

(4) Compute for $\mathbb{K} = \mathbb{Z}$, $\mathbb{G} = \mathbb{Z}/6$ the homology groups $H_*(\mathbb{R}P^4; \mathbb{Z}/6)$, starting with your knowledge of $H_*(\mathbb{R}P^4; \mathbb{Z})$.

Aufgabe 5. (Künneth Theorem) [3+1+6 points]

Let $\mathbb{K}$ be a principal ideal domain.

(1) Formulate the Künneth Theorem for the homology of free chain complexes.

(2) Formulate the Künneth Theorem for the homology of spaces.

(3) For $\mathbb{K} = \mathbb{Z}$, compute $H_*(\mathbb{R}P^2 \times \mathbb{C}P^2; \mathbb{Z})$ in all degrees, starting with your knowledge of $H_*(\mathbb{R}P^2; \mathbb{Z})$ and $H_*(\mathbb{C}P^2; \mathbb{Z})$.

Aufgabe 6. (Self-maps degrees) [1+1+4+4 points]

Let us abbreviate $H_*(\quad) = H_*(\quad;\mathbb{Z})$ and $H^*(\quad) = H^*(\quad;\mathbb{Z})$. Assume X is a space such that each homology group $H_n(X)$ is either trivial or free of rank 1, generated by some element c_n.

(1) Then each cohomology group $H^n(X)$ is also either trivial or free of rank 1. Denote a generator by γ_n.

(2) For any self-map $f\colon X \to X$ we define (for each degree n, where $H_n(X)$ and $H^n(X)$ are non-trivial) natural numbers λ_n by the equations

$$f_*(c_n) = \lambda_n \, c_n \, .$$

Show that the equations

$$f^*(\gamma_n) = \lambda_n \, \gamma_n$$

follow.

(3) Using the action of $H^\#(X) := \bigoplus_q H^q(X)$ on $H_\#(X) := \bigoplus_n H_n(X)$ via the cap-product we define natural numbers $\mu_{q,n}$ by the equations

$$\gamma_q \cap c_n = \mu_{q,n} \, c_{n-q},$$

for degrees $q, n, n-q$ such that $H^q(X)$, $H_n(X)$ and $H_{n-q}(X)$ are non-trivial. Use the adjunction formula $f_*(f^*(\gamma_q) \cap c_n) = \gamma_q \cap f_*(c_n)$ to prove the equations

$$\lambda_q \, \lambda_{n-q} \, \mu_{q,n} = \lambda_n \, \mu_{q,n}$$

and conclude

$$\lambda_q \, \lambda_{n-q} = \lambda_n,$$

when $\mu_{q,n} \neq 0$.

(4) Now assume, that $X = M$ is an oriented, connected, compact manifold without boundary and of dimension m. Choose c_m to be the fundamental class. Prove, that for all q, for which $H^q(M) \cong H_{m-q}(M)$ is non-trivial,

$$\mu_{q,m} = \pm 1 \, .$$

Conclude in the case $m = 2n$, $q = n$ and $H^n(M) \cong H_n(M)$ non-trivial that

$$\lambda_q{}^2 = \lambda_m \, .$$

Aufgabe 7. (Derived Functors) [2+2+2+4 points]

Let $F\colon \mathbb{K}-\mathrm{MOD} \to \mathbb{K}-\mathrm{MOD}$ be an additive covariant functor from the category of $\mathbb{K}$-modules to itself.

(1) When do we call F left-exact, right-exact and exact ?

(2) Extend F to a functor from the category $\partial\mathbb{K}-\mathrm{MOD}$ of chain complexes over $\mathbb{K}$ to itself.

(3) What is a projective resolution of a module $\mathbb{M}$ over $\mathbb{K}$?

(4) Define the left derived functors L_iF of a right-exact additive functor F.

Aufgabe 8. (Homology and cohomology in top dimensions) [3+7 points]

Abbreviate $H_*(\quad;\mathbb{Z})$ by $H_*(\quad)$ and $H^*(\quad;\mathbb{Z})$ by $H^*(X)$.

(1) Prove or disprove: If X is a CW complex of finite type and of dimension m, then $H^q(X) = 0$ for $q > m$.

(2) Prove for a CW complex X of finite type and dimension m, that $H_m(X)$ is free (or trivial). Conclude that $H^m(X)$ contains a free summand of the same rank.

Aufgabe 9. (Poincaré Duality) [3+1+1+2+3 points]

(1) What is an orientation of a manifold M of dimension m ?

(2) What is a fundamental class of M ?

(3) Define the Poincaré duality homomorphism $\mathrm{PD}_M \colon H^q(M;\mathbb{Z}) \to H_{m-q}(M,\mathbb{Z})$.

(4) When is PD_M an isomorphism ?

(5) For which maps $f \colon M \to M'$ between two orientable, connected and compact manifolds M and M' of the same dimension m does the formula

$$f_*(\mathrm{PD}_M(f^*(\beta))) = \mathrm{PD}_{M'}(\beta)$$

hold for all $\beta \in H^*(M')$?

Aufgabe 10. (The Pontryagin product: a new product) [2+2+2+4 points]

— **Activate your mathematical imagination !** —

Let G be a Lie group of dimension m; denote the product by $\mu\colon G \times G \to G$. We define a new product, called the *Pontryagin product* on the homology $H_*(G) = H_*(G;\mathbb{Z})$ of G, as follows

$$\bullet\colon H_i(G) \otimes H_j(G) \xrightarrow{\times} H_{i+j}(G \times G) \xrightarrow{\mu_*} H_{i+j}(G)$$

that is, as the composition of the homology cross product and the homomorphism induced by the multiplication μ in the Lie group G.

(1) Show that the product is associative: $a \bullet (b \bullet c) = (a \bullet b) \bullet c$ for all $a, b, c \in H_*(G)$.

(2) Show that the base class $[e] \in H_0(X)$, represented by the neutral element e in the Lie group G, is the neutral element: $[e] \bullet a = a \bullet [e] = a$ for all $a \in H_*(G)$.

(3) If the Lie group G is abelian, show that the product is graded commutative: $a \bullet b = (-1)^{ij} b \bullet a$ for $a \in H_i(G)$ and $b \in H_j(b)$.

(4) Example: One can quickly see, that for $G = \mathbb{S}^3 = \mathrm{SU}(2)$ all products of classes in positive degrees are trivial. But consider the r-dimensional torus $G = (\mathbb{S}^1)^r$, whose homology is $H_*(G) \cong \Lambda_{\mathbb{Z}}[x_1, \ldots, x_r]$, an exterior algebra over $\mathbb{Z}$ on r variables in degree 1. For $r = 2$, find the multiplication table for the new product.

Zusatzaufgabe [1 + 1 bonus points]

Which of the following mathematicians were no topologists ? (Delete their names.)

Poincaré
Brouwer
Seifert
Hurwitz
Alexander
Lefschetz
Steenrod
H. Hopf
E. Hopf
Eilenberg
MacLane
Künneth
Hurewicz
Whitney

Viel Erfolg !