Aufgaben zur Topologie II

Prof. Dr. C.-F. Bödigheimer

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Week 13 — Cup products, Hopf invariant and acyclic models no due date

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Exercise 13.1 (Support of a cohomology class)
For a cohomology class $\alpha \in H^i(X)$ with $i > 0$ we say *its support is in a subset* $A \subset X$ if $\alpha \in \ker(r^*\colon H^i(X) \to H^i(X - A))$ for the inclusion of the complement $r\colon X - A \to X$.

(1) Show the following: If α has support in $A \subset X$ and β has support in $B \subset X$, and $A \cap B = \emptyset$, and A, B are closed, then $\alpha \cup \beta = 0$.

(2) Application: If X is an orientable surface of genus g, we do know a symplectic basis $a_1, \ldots, a_g, b_1, \ldots, b_g$ of $H_1(X;\mathbb{Z}) \cong \mathbb{Z}^{2g}$, given by simply-closed curves a_i and b_j in X, such that each curve intesects (after a homotopy) exactly one other curve exactly once, namely a_i and b_i. The Universal Coefficient Theorem $H^1(X;\mathbb{Z}) \cong \mathrm{Hom}(H_1(X;\mathbb{Z}),\mathbb{Z})$ and this basis allows us to associate to these curves a basis $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ of the cohomology $H^1(X;\mathbb{Z}) \cong \mathbb{Z}^{2g}$. Then $\alpha_i \cup \alpha_j = 0$ and $\beta_i \cup \beta_j = 0$ for all i and j, and $\alpha_i \cup \beta_j = 0$ if $i \neq j$. Thus the cup products vanish if the corresponding curves do not intersect.

Exercise 13.2 (Intersections)
But what if they do intersect? Study this situation, using Exerc. 13.1, starting with the torus and using Exerc. 13.3.

Exercise 13.3 (Cohomology ring of a product)
Let $\mathbb{F}$ be a field and write $H^\#(X;\mathbb{F}) := \bigoplus_{i\geq 0} H^i(X;\mathbb{F})$ for the graded cohomology ring of X. Using the Künneth Formula we have an isomorphism

$$H^\#(X_1 \times X_2;\mathbb{F}) \cong H^\#(X_1;\mathbb{F}) \otimes H^\#(X_2;\mathbb{F})$$

of graded vector spaces. Show that this is an isomorphism of graded rings by showing the formula:

$$(\alpha_1 \times \alpha_2) \cup (\beta_1 \times \beta_2) = (\alpha_1 \cup \alpha_2) \times (\beta_1 \cup \beta_2)$$

for $\alpha_1, \beta_1 \in H^\#(X_1;\mathbb{F})$ and $\alpha_2, \beta_2 \in H^\#(X_2;\mathbb{F})$.

Exercise 13.4 (Hopf Invariant)
For a map $f\colon \mathbb{S}^{2n-1} \to \mathbb{S}^n$ with $n \geq 1$ consider the mapping cone $C_f := \mathbb{S}^n \cup_f \mathbb{D}^{2n}$, attaching a $2n$-cell to an n-sphere via f.
a) Compute the cohomology groups of C_f using the long exact sequence associated to the Puppe sequence

$$\mathbb{S}^{2n-1} \xrightarrow{f} \mathbb{S}^n \xrightarrow{i} C_f \xrightarrow{p} \Sigma\mathbb{S}^{2n} \xrightarrow{\Sigma(f)} \Sigma\mathbb{S}^n$$

to find $H^n(C_f) \cong \mathbb{Z}$, generated by some u with $i^*(u) = o_n$, and $H^{2n}(C_f) \cong \mathbb{Z}$, generated by $v = p^*(o_{2n})$, where i and p are inclusion into the mapping cone resp. the projection from the mapping cone in the sequence above and o_n and o_{2n} are the standard generators of the cohomology of the spheres $\mathbb{S}^n$ resp. $\mathbb{S}^{2n}$.
Famous example: The Hopf map $\eta\colon \mathbb{S}^3 \to \mathbb{S}^2$, given for $(z_1, z_2) \in \mathbb{S}^3 \subset \mathbb{C}^2$ by $\eta(z_1, z_2) = [z_1 : z_2] = z_1/z_2$ in homogenous notation in $\mathbb{C}P^1 \simeq \mathbb{S}^2$ and as a ratio in $\mathbb{C} \cup \{\infty\} = \mathbb{S}^2$. Note that the mapping cone C_η is the complex projective plane $\mathbb{C}P^2$.

The cohomology element $u^2 = u \cup u$ lies in $H^{2n}(C_f)$ and is therefore some multiple of v; we define the natural number $\mathbf{h}(f)$ by the equation

$$u^2 = \mathbf{h}(f)\, v$$

18.1.1 Properties of the cap product.

(1) For $f\colon (X; A, B) \to (X'; A', B')$, $x' \in H^p(X', A'; M)$, and for $u \in H_{p+q}(X, A \cup B; N)$ the relation $f_*(f^*x' \cap u) = x' \cap f_*u$ holds.

(2) Let A, B be excisive, $j_B\colon (B, A \cap B) \to (X, A \cup B)$ the inclusion and

$$\partial_B\colon H_{p+q}(X, A \cup B) \xrightarrow{\partial} H_{p+q-1}(A \cup B, A) \xleftarrow{\cong} H_{p+q-1}(B, A \cap B).$$

Then for $x \in H^p(X, A; M)$, $y \in H_{p+q}(X, A \cup B; N)$,

$$j_B^* x \cap \partial_B y = (-1)^p \partial(x \cap y) \in H_{q-1}(B; M \otimes N).$$

(3) Let A, B be excisive, $j_A\colon (A, A \cap B) \to (X, B)$ the inclusion and

$$\partial_A\colon H_{p+q}(X, A \cup B) \xrightarrow{\partial} H_{p+q-1}(A \cup B, B) \xleftarrow{\cong} H_{p+q-1}(A, A \cap B).$$

Then for $x \in H^p(A; M)$, $y \in H_{p+q}(X, A \cup B; N)$,

$$j_{A*}(x \cap \partial_A y) = (-1)^{p+1} \delta x \cap y \in H_{q-1}(X, B; M \otimes N).$$

(4) $1 \cap x = x$, $1 \in H^0(X)$, $x \in H_n(X, B)$.

(5) $(x \cup y) \cap z = x \cap (y \cap z) \in H_{n-p-q}(X, C; \Lambda)$ for $x \in H^*(X, A; R)$, $y \in H^*(X, B; \Lambda)$, $z \in H_*(X, A \cup B \cup C; \Lambda)$.

Figure 1: From Tammo tom Dieck, Algebraic Topology

and call it the *Hopf invariant* of f. Prove the following assertions:

b) If $f \simeq f'$, the $\mathbf{h}(f) = \mathbf{h}(f')$.
c) If f is null-homotopic, then $\mathbf{h}(f) = 0$.
d) $\mathbf{h}(\varphi \circ f) = \deg(\varphi)^2\,\mathbf{h}(f)$ for any $\varphi\colon \mathbb{S}^n \to \mathbb{S}^n$.
e) $\mathbf{h}(f \circ \psi) = \deg(\psi)\,\mathbf{h}(f)$ for any $\psi\colon \mathbb{S}^{2n} \to \mathbb{S}^{2n}$.
f)* Prove that, in the famous example above, $\mathbf{h}(\eta) \neq 0$. Deduce that $\pi_3(S^2)$ is not trivial (actually, it is isomorphic to $\mathbb{Z}$ generated by the Hopf map η).

Exercise 13.5* (Acyclic Model Theorem)
Let $\mathcal{C}$ be any category and let $\mathcal{D} = \partial\mathbb{Z} - \mathrm{MOD}^+$ be the category of augmented chain complexes of abelian groups. We consider functors $F\colon \mathcal{C} \to \mathcal{D}$ which are acyclic or free with respect to a given family $\mathcal{M}$ of objects in $\mathcal{C}$, meaning the following.
a) F is called *acyclic* with respect to $\mathcal{M}$ if $F(M)$ is acyclic for all $M \in \mathcal{M}$.
b) F is called *free in dimension* n with respect to $\mathcal{M}$ if one has the following:

- an index set J_n;
- a family $(M_\alpha \mid \alpha \in J_n)$ of objects in $\mathcal{M}$;
- an element $b_\alpha \in F(M_\alpha)_n$ for each $\alpha \in J_n$,

such that for each object X in $\mathcal{C}$, a basis of $F(X)_n$ is given by the set of elements $\beta_{\alpha,\varphi}$ for α varying in J_n and φ varying in $\mathrm{mor}_{\mathcal{C}}(M_\alpha, X)$; the element $\beta_{\alpha,\varphi} \in F(X)_n$ is defined as the image of $b_\alpha \in F(M_\alpha)_n$ under the map $F(\varphi)$. F is called *free* if it is free in each dimension.

Example: Let $\mathcal{C}$ be the category of topological spaces and continuous maps, and let $F(X)$ be the singular chain complex of a space. This F is an acyclic functor with respect to any family $\mathcal{M}$ of spaces. If we take $\mathcal{M}$ to be all simplices Δ^n, the index set $J_n = \{n\}$ a singelton, $M_n = \Delta^n$ and $b_n = \mathrm{id}_{\Delta^n} \in S_n(\Delta^n)$, then F is free with respect to $\mathcal{M}$.

Now one can prove the following **Acyclic Model Theorem**:

Let $F, F' \colon \mathcal{C} \to \partial\mathbb{Z} - \mathrm{MOD}^{+}$ two functors and assume that F is free and F' is acyclic with respect to some family $\mathcal{M}$ of objects in $\mathcal{C}$. Then
(i) There is a natural transformation $\theta \colon F \to F'$.
(ii) For any two natural transfomations $\theta, \theta' \colon F \to F'$ there is a natural chain homotopy between them.

Hint: If such a θ exists, in particular it gives a map $\theta(M_\alpha)_n \colon F(M_\alpha)_n \to F'(M_\alpha)_n$ for all $\alpha \in J_n$, and in particular one can define $b'_\alpha := \theta_n(M_\alpha)(b_\alpha) \in F'(M_\alpha)_n$. Show that, conversely, for any choices of such elements $b'_\alpha \in F'(M_\alpha)_n$ there is a natural transformation $\theta \colon F \to F'$.
See J.R. Munkres, Elements of Algebraic Topology, pp. 183-185, for help with the proof.