Aufgaben zur Topologie II

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Week 12 — Homology cross product and the Künneth theorem again due by: 19.7.2017

Exercise 12.1 (Cellular chain complexes of products)
Let X and Y be two finite CW-complexes and denote respectively by $\mathcal{C}_\bullet(X)$ and $\mathcal{C}_\bullet(Y)$ their cellular chain complexes. Show that the product $X \times Y$ has a CW-structure such that there is an isomorphism $\mathcal{C}_\bullet(X \times Y) \cong \mathcal{C}_\bullet(X) \otimes \mathcal{C}_\bullet(Y)$, in contrast to the situation of the Alexandrer-Whitney and the Eilenberg-Zilber map for the singular chain complexes.

Exercise 12.2 (Homology with coefficients and Künneth theorem)
Let $\mathbb{K}$ be a PID and $\mathbb{G}$ be a module over $\mathbb{K}$. Prove the following isomorphisms by comparing the universal coefficient theorem and the Künneth theorem:

(1) $H_n(X \times Y; \mathbb{G}) \cong \oplus_{i+j=n} H_i(X; H_j(Y; \mathbb{G}))$;

(2) $H^n(X \times Y; \mathbb{G}) \cong \oplus_{i+j=n} H^i(X; H^j(Y; \mathbb{G}))$.

Proposition 4.1. *Let C', C, C'' be chain complexes of abelian groups. Then there is a natural isomorphism*

$$(C' \otimes C) \otimes C'' \cong C' \otimes (C \otimes C'') . \quad \square \tag{4.1}$$

We are going to exploit (4.1) together with the companion formula (which is just Theorem 1.1 in the case $\Lambda = \mathbb{Z}$)

$$\mathrm{Hom}(C' \otimes C, C'') \cong \mathrm{Hom}(C', \mathrm{Hom}(C, C'')) . \tag{4.2}$$

First, we consider (4.1). We take C', C, C'' to be resolutions of abelian groups A', A, A''. Thus, for example $C_1 = R$, $C_0 = F$, $C_p = 0$, $p \neq 0, 1$, and ∂_1 is the inclusion $R \subseteq F$, where

$$0 \to R \to F \to A \to 0$$

is a free presentation of A. If we compute homology by means of the Künneth formula on either side of (4.1), we find

$$H_0((C' \otimes C) \otimes C'') = (A' \otimes A) \otimes A'' ,$$
$$H_1((C' \otimes C) \otimes C'') \cong \mathrm{Tor}(A', A) \otimes A'' \oplus \mathrm{Tor}(A' \otimes A, A'') ,$$
$$H_2((C' \otimes C) \otimes C'') = \mathrm{Tor}(\mathrm{Tor}(A', A), A'') ;$$
$$H_0(C' \otimes (C \otimes C'')) = A' \otimes (A \otimes A'') ,$$
$$H_1(C' \otimes (C \otimes C'')) \cong A' \otimes \mathrm{Tor}(A, A'') \oplus \mathrm{Tor}(A', A \otimes A'') ,$$
$$H_2(C' \otimes (C \otimes C'')) = \mathrm{Tor}(A', \mathrm{Tor}(A, A'')) ,$$

where Tor means $\mathrm{Tor}_1^{\mathbb{Z}}$. We readily infer

Theorem 4.2. *Let A', A, A'' be abelian groups. There is then an unnatural isomorphism*

$$\mathrm{Tor}(A', A) \otimes A'' \oplus \mathrm{Tor}(A' \otimes A, A'') \cong A' \otimes \mathrm{Tor}(A, A'') \oplus \mathrm{Tor}(A', A \otimes A'') , \tag{4.3}$$

and a natural isomorphism

$$\mathrm{Tor}(\mathrm{Tor}(A', A), A'') \cong \mathrm{Tor}(A', \mathrm{Tor}(A, A'')) . \tag{4.4}$$

From Hilton–Stambach: A Course in Homological Algebra.

Exercise 12.3 (Multiplicativity of Euler characteristic, Betti numbers and Poincaré polynomial)
Let X and Y be two spaces of finite type; prove the following equalities using the Künneth sequence:

(1) $\chi(X \times Y) = \chi(X)\chi(Y)$,

(2) $b_n(X \times Y) = \sum_{i=0}^{n} b_i(X) b_{n-i}(Y)$,

(3) $P_t(X \times Y) = P_t(X) P_t(Y)$.

We now turn to (4.2) and use the same chain complexes $\boldsymbol{C}', \boldsymbol{C}, \boldsymbol{C}''$ as in the proof of Theorem 4.2. Computing either side of (4.2) by means of the dual Künneth formula, we find

$$H_0(\mathrm{Hom}(\boldsymbol{C}' \otimes \boldsymbol{C}, \boldsymbol{C}'')) = \mathrm{Hom}(A' \otimes A, A''),$$
$$H_{-1}(\mathrm{Hom}(\boldsymbol{C}' \otimes \boldsymbol{C}, \boldsymbol{C}'')) = \mathrm{Hom}(\mathrm{Tor}(A', A), A'') \oplus \mathrm{Ext}(A' \otimes A, A''),$$
$$H_{-2}(\mathrm{Hom}(\boldsymbol{C}' \otimes \boldsymbol{C}, \boldsymbol{C}'')) = \mathrm{Ext}(\mathrm{Tor}(A', A), A'');$$
$$H_0(\mathrm{Hom}(\boldsymbol{C}', \mathrm{Hom}(\boldsymbol{C}, \boldsymbol{C}''))) = \mathrm{Hom}(A', \mathrm{Hom}(A, A'')),$$
$$H_{-1}(\mathrm{Hom}(\boldsymbol{C}', \mathrm{Hom}(\boldsymbol{C}, \boldsymbol{C}''))) = \mathrm{Hom}(A', \mathrm{Ext}(A, A'')) \oplus \mathrm{Ext}(A', \mathrm{Hom}(A, A'')),$$
$$H_{-2}(\mathrm{Hom}(\boldsymbol{C}', \mathrm{Hom}(\boldsymbol{C}, \boldsymbol{C}''))) = \mathrm{Ext}(A', \mathrm{Ext}(A, A'')),$$

where Tor means $\mathrm{Tor}_1^{\mathbb{Z}}$ and Ext means $\mathrm{Ext}_{\mathbb{Z}}^1$. We readily infer, leaving all details to the reader,

Theorem 4.3. *Let A', A, A'' be abelian groups. There is then an unnatural isomorphism*

$$\begin{aligned} &\mathrm{Hom}(\mathrm{Tor}(A', A), A'') \oplus \mathrm{Ext}(A' \otimes A, A'') \\ &\quad \cong \mathrm{Hom}(A', \mathrm{Ext}(A, A'')) \oplus \mathrm{Ext}(A', \mathrm{Hom}(A, A'')), \end{aligned} \tag{4.5}$$

and a natural isomorphism

$$\mathrm{Ext}(\mathrm{Tor}(A', A), A'') \cong \mathrm{Ext}(A', \mathrm{Ext}(A, A'')). \quad \square \tag{4.6}$$

From Hilton–Stambach: A Course in Homological Algebra.

Exercise 12.4 (Adjoint functors)
A pair of functors $L\colon \mathcal{A} \to \mathcal{B}$ and $R\colon \mathcal{B} \to \mathcal{A}$ are called *adjoint* and more precisely L is called a *left-adjoint* and R is called a *right-adjoint* if for all objects $A \in \mathcal{A}$ and all $B \in \mathcal{B}$, there is a natural bijection

$$\tau = \tau_{A,B}\colon\ \mathrm{mor}_{\mathcal{B}}(L(A), B) \longrightarrow \mathrm{mor}_{\mathcal{A}}(A, R(B))$$

of morphism sets.

(0) What does naturality mean in this definition ?

(1) With $\mathcal{A} = \mathbb{K}$–MOD, $\mathcal{B} =$ SETS, the functor $R(A) = A$, the forgetful functor, and $L(B) = \mathrm{Fr}_{\mathbb{K}}(B) =$ the free $\mathbb{K}$-module generated by the set B, are an adjoint pair.

(2) With $\mathcal{A} = \mathcal{B} = \mathbb{K}$–MOD and a fixed $\mathbb{K}$-module M, the functors $L(A) = A \otimes_{\mathbb{K}} M$ and $R(B) = \mathrm{Hom}_{\mathbb{K}}(M, B)$ are an adjoint pair.

Now we show that a left-adjoint it right-exact and that a right-adjoint is left-exact in case $\mathcal{A} = \mathcal{B} = \mathbb{K}$–MOD.

(3) Consider $\mathbb{K}$-modules A, B and C together with two homomorphisms $f\colon A \to B$ and $g\colon B \to C$. Show that $A \xrightarrow{f} B \xrightarrow{g} C$ is exact at B if the induced sequence

$$\mathrm{Hom}(M, A) \xrightarrow{f_*} \mathrm{Hom}(M, B) \xrightarrow{g_*} \mathrm{Hom}(M, C)$$

is exact at $\mathrm{Hom}(M, B)$ for every $\mathbb{K}$-module M.

(3') Formulate the analogous statement for the contravariant Hom-functor.

(4) Let $L, R\colon \mathbb{K}$–MOD $\to \mathbb{K}$–MOD be an adjoint pair of additive functors. Show that L is right-exact and that R is left-exact.

Hint: Use exercises 7.4, 7.5 and part (3).

Exercise 12.5* (Dual Künneth Theorem)
Let C and D be chain complexes over a PID $\mathbb{K}$ and assume that C is free. Then there is a natural short exact sequence

$$0 \longrightarrow \prod_{q-p=n+1} \mathrm{Ext}_{\mathbb{K}}(H_p(C), H_q(D)) \longrightarrow H_n(\mathrm{Hom}_{\mathbb{K}}(C, D)) \longrightarrow \prod_{q-p=n} \mathrm{Hom}_{\mathbb{K}}(H_p(C), H_q(D)) \longrightarrow 0$$

Its splits unnaturally. The second map sends a chain map $f\colon C \to D$ of degree n, if it is a cycle in $\mathrm{Hom}_{\mathbb{K}}(C, D)$, to the map $f_* = H_*(f) : H_p(C) \to H_q(D)$ it induces in homology.

Corollary 4.5. *If A' is torsion-free, then* $\mathrm{Ext}(A', \mathrm{Ext}(A, A'')) = 0$ *for all* A, A''. □

Corollary 4.6. (i) *There is a natural isomorphism*

$$\mathrm{Ext}(A', \mathrm{Ext}(A, A'')) \cong \mathrm{Ext}(A, \mathrm{Ext}(A', A'')).$$

(ii) *There is an unnatural isomorphism*

$$\mathrm{Hom}(A', \mathrm{Ext}(A, A'')) \oplus \mathrm{Ext}(A', \mathrm{Hom}(A, A''))$$
$$\cong \mathrm{Hom}(A, \mathrm{Ext}(A', A'')) \oplus \mathrm{Ext}(A, \mathrm{Hom}(A', A'')). \quad \square$$

From Hilton–Stambach: A Course in Homological Algebra.