Aufgaben zur Topologie II

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Week 11 — Derived Functors, Künneth Theorem due by: 12.7.2017

Exercise 11.1 ($\lim^0$ and $\lim^1$.)
Consider for a sequence $\mathfrak{A} : \ldots \to A_n \xrightarrow{\varphi_n} A_{n-1} \ldots \to A_0$ of $\mathbb{K}$-modules the induced 'shift' homomorphism

$$\Phi \colon \prod A_i \longrightarrow \prod A_i, \qquad \Phi(\ldots, a_n, a_{n-1}, \ldots, a_0) = (\ldots, \varphi_{n+1}(a_{n+1}), \varphi_n(a_n), \ldots, \varphi_1(a_1)).$$

Recall the definition of the *(inverse) limit* of $\mathfrak{A}$ as $\lim^0 \mathfrak{A} = \lim \mathfrak{A} := \ker(\mathrm{id} - \Phi)$ and the *derived (inverse) limit* as $\lim^1 \mathfrak{A} := \operatorname{coker}(\mathrm{id} - \Phi)$.

1. The snake lemma says: $\lim^0$ is left-exact and $\lim^1$ is right-exact as functors from the additive category of inverse systems (over the ordered set $\mathbb{N}$) of $\mathbb{K}$-modules to the category of $\mathbb{K}$-modules.
2. $\lim^1(\mathfrak{A}) \cong R^1(\lim^0)(\mathfrak{A})$ is the first right-derived functor of $\lim^0$.
3. $\lim^n(\mathfrak{A}) := R^n(\lim^0)(\mathfrak{A}) = 0$ for $n \geq 2$.
4. Let $A_0 = A$ be a group and all $\varphi_i \colon A_i \subset A_{i-1}$ be subgroups of A; in this case $\lim \mathfrak{A} = \bigcap_i A_i$ is the intersection. Assume further, that A is a topological group and all cosets $a + A_i$ are open.
   (i) $\lim^0 \mathfrak{A} = 0$ iff A is hausdorff.
   (ii) $\lim^1 \mathfrak{A} = 0$ iff A is complete (i.e., every Cauchy sequence has a limit, possibly not unique).

Exercise 11.2 (Projective modules)
For a $\mathbb{K}$-module P the following statements are equivalent:

(i) P is projective.

(ii) For every short exact sequence $0 \to A \to B \to C \to 0$ of $\mathbb{K}$-modules the induced sequence

$$0 \to \mathrm{Hom}_{\mathbb{K}}(P, A) \to \mathrm{Hom}_{\mathbb{K}}(P, B) \to \mathrm{Hom}_{\mathbb{K}}(P, C) \to 0$$

is exact.

(iii) If $\pi \colon B \to P$ is a surjection, there is a section $s \colon P \to B$ with $\pi \circ s = \mathrm{id}_P$.

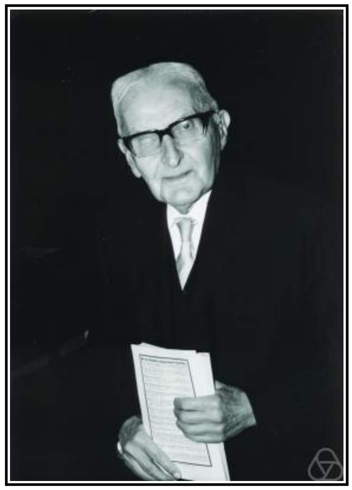
(iv) P is a direct summand of every module of which it is a quotient.

(v) P is a direct summand of some free module.

Exercise 11.3 (Injective Modules over Principal Ideal Domains)
A module over a principal ideal domain is injective if and only if it is divisible.

(Hint: To show divisibility of $x \in M$ by a scalar $0 \neq \lambda \in \mathbb{K}$ consider the multiplication by $\lambda \colon \mathbb{K} \to \mathbb{K}$ and the homomorphism $\psi \colon \mathbb{K} \to M$ given by $\psi(1) = x$. Vice versa, given a monomorphism $\iota \colon A \subset B$ and a homomorphism $\psi \colon A \to M$, to extend ψ to B consider all pairs (A_i, ψ_i) with $A \leq A_i \leq B$ an intermediate submodule and $\psi_i \colon A_i \to M$ a partial extension of ψ. They form an ordered set, to which one can apply Zorn's Lemma. To show $\bar{A} = B$ for a maximal element $(\bar{A}, \bar{\psi})$ you will finally need that $\mathbb{K}$ is a PID.)

Exercise 11.4 (Künneth Theorem.)
Compute the homology and cohomology with coefficients in $\mathbb{Z}$ and $\mathbb{Z}/2$ for the following product spaces:



Hermann Künneth, 6. Juli 1892 — 7. Mai 1975.

1. $(\mathbb{S}^1)^n$, a torus of dimension n.
2. $\mathbb{S}^2 \times \mathbb{R}P^3$, the product of a 2-sphere and the projective space.
3. $\mathbb{R}P^2 \times \mathbb{R}P^3$, the product of the projective plane and the projective space.

Exercise 11.5* (Universal Coefficient Theorem expressing homology via cohomology)
The following two exercises are from S. MacLane: Homology, p. 172.

1. For abelian groups G and A construct natural homomorphisms $\mathrm{Hom}(G,\mathbb{Z}) \otimes A \to \mathrm{Hom}(G,A)$ and $G \otimes A \to \mathrm{Hom}(\mathrm{Hom}(G,\mathbb{Z}),A)$. Shom them isomorphisms when G is a finitely generated free group. Show them chain transformations when G is a chain complex.
2. Let K be a complex of abelian groups, with each K_n a finitely generated free abelian group. Write $H^n(K)$ for $H^n(\mathrm{Hom}(K,\mathbb{Z}))$. Using Ex. 1 and the universal coefficient theorems, establish natural exact sequences

$$0 \longrightarrow H^n(K) \otimes G \longrightarrow H^n(\mathrm{Hom}(K,G)) \longrightarrow \mathrm{Tor}(H^{n+1}(K),G) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Ext}(H^{n+1}(K),A) \longrightarrow H_n(K \otimes A) \longrightarrow \mathrm{Hom}(H^n(K),A) \longrightarrow 0.$$