Aufgaben zur Topologie II

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Week 10 — Universal Coefficient Theorems due by: 5.7.2017

Exercise 10.1 (Euler characteristic over different fields)
Let X be of finite type, so that the Euler characteristic $\chi(X) := \sum_{i\geq 0}(-1)^i \operatorname{rk}(H_i(X;\mathbb{Z}))$ is defined. Prove that for any field $\mathbb{F}$ the following equality holds

$$\chi(X) = \sum_{i\geq 0}(-1)^i \dim_{\mathbb{F}}(H_i(X;\mathbb{F})).$$

> **7. Torsion Products of Modules**
>
> For fixed $n \geqq 0$ we consider chain complexes L of length n,
>
> $$L\colon\ L_0 \xleftarrow{\partial} L_1 \xleftarrow{\partial} \cdots \xleftarrow{\partial} L_{n-1} \xleftarrow{\partial} L_n,$$
>
> with each L_k a finitely generated projective right R-module. The dual $L^* = \mathrm{Hom}_R(L,\mathrm{R})$ can also be regarded as a chain complex L^*, with L_k^* as the chains of dimension $n-\mathrm{k}$,
>
> $$L^*\colon\ L_n^* \xleftarrow{\delta} L_{n-1}^* \xleftarrow{\delta} \cdots \xleftarrow{\delta} L_1^* \xleftarrow{\delta} L_0^*.$$

From Mac Lane: Homology, Chapter 5.7.

Exercise 10.2 (Moore spaces)
Let $X_n := \mathbb{S}^1 \cup_f e^2$ be a Moore space for a map $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ of degree n. For example, $X_2 = \mathbb{R}P^2$, the real projective plane. Compute the homology groups $H_*(X_n;\mathbb{Z}/m)$ for $n = 0, 2, 4$ and $m = 0, 2, 4$.

> If G is a right R-module, regarded as a trivial chain complex, a chain transformation $\mu\colon L \to G$ is a module homomorphism $\mu_0\colon L_0 \to G$ with $\mu_0\partial = 0\colon L_1 \to G$, while a chain transformation $\nu\colon L^* \to {}_R C$ is a module homomorphism $\nu\colon L_n^* \to C$ with $\nu\,\delta = 0$. For given modules G_R and ${}_R C$ we take as the elements of $\mathrm{Tor}_n^R(G,C)$ all the triples
>
> $$t = (\mu, L, \nu), \quad \mu\colon L \to G, \quad \nu\colon L^* \to C,$$
>
> where L has length n and μ, ν are chain transformations, as above.

From Mac Lane: Homology, Chapter 5.7.

Exercise 10.3 (Homology with coefficients)

1. Let $f\colon X \to Y$ be a continous map and assume that $f_*\colon H_n(X;\mathbb{Z}) \xrightarrow{\cong} H_n(Y;\mathbb{Z})$ is an isomorphism for all $n \geq 0$. Then for all coefficient groups $\mathbb{G}$, we have isomorphisms

$$f_*\colon H_n(X;\mathbb{G}) \xrightarrow{\cong} H_n(Y;\mathbb{G})$$

   and

$$f^*\colon H^n(Y;\mathbb{G}) \xrightarrow{\cong} H^n(X;\mathbb{G})$$

   for all $n \geq 0$. (What if the assumption holds only in degrees n and $n-1$ for a specific n ?)

2. If X is a finite CW complex, then the homology $H_*(X;\mathbb{G})$ and cohomology $H^*(X;\mathbb{G})$ is of finite type for any finitely generated coefficient module $\mathbb{G}$.

3. If X is a finite-dimensional CW complex, above which dimension does the homology $H_*(X;\mathbb{G})$ and cohomology $H^*(X;\mathbb{G})$ certainly vanish for any coefficient module $\mathbb{G}$?

4. If X is a space with homology grops $H_n(X;\mathbb{Z})$ vanishing for $n > d$, in which dimensions does the homology $H_n(X;\mathbb{G})$ and cohomology $H^*(X;\mathbb{G})$ certainly vanish for any coefficient group $\mathbb{G}$? Find — by looking at other exercises of this sheet — an example with non-vanishing homology or cohomology in degree $d+1$.

5. If X is a space with vanishing homology groups $H_n(X;\mathbb{Z})$ for $0 < n < c$, in which dimensions does the homology $H_n(X;\mathbb{G})$ and cohomology $H*(X;\mathbb{G})$ certainly vanish for any coefficient group $\mathbb{G}$?

> **Proposition 7.1.** The symbols (μ, L, ν) in Tor, are additive in μ and ν; *e.g.,*
>
> $$(\mu_1 + \mu_2, L, \nu) = (\mu_1, L, \nu) + (\mu_2, L, \nu). \qquad (7.4)$$

From Mac Lane: Homology, Chapter 5.7.

Exercise 10.4 (Tor and torsion)
For $\mathbb{Z}$-modules $\mathbb{A}$ and $\mathbb{B}$ (i.e. abelian groups) we have

1. $\operatorname{Tor}_{\mathbb{Z}}(\mathbb{A},\mathbb{B}) \cong \operatorname{Tor}_{\mathbb{Z}}(\operatorname{tors}(\mathbb{A}),\operatorname{tors}(\mathbb{B}))$ using the exact sequence $0 \to \operatorname{tors}(\mathbb{A}) \to \mathbb{A} \to \mathbb{A}/\operatorname{tors}(\mathbb{A}) \to 0$ and the same sequence for $\mathbb{B}$.

2. Use the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ to get $\operatorname{tors}(\mathbb{A}) \cong \operatorname{Tor}_{\mathbb{Z}}(\mathbb{A},\mathbb{Q}/\mathbb{Z})$ and $\operatorname{fr}(\mathbb{A}) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \cong \mathbb{A} \otimes \mathbb{Q}/\mathbb{Z}$, where $\operatorname{fr}(\mathbb{A}) := \mathbb{A}/\operatorname{tors}(\mathbb{A})$ is the largest torsion-free quotient of $\mathbb{A}$.

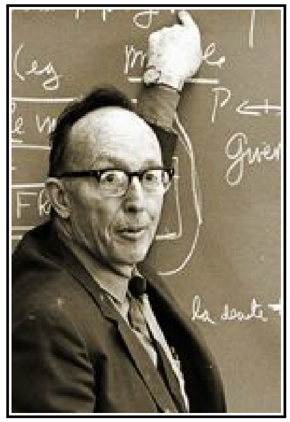
3. Deduce the exact sequence

$$0 \to H_n(X,A;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \to H_n(X,A;\mathbb{Q}/\mathbb{Z}) \to \operatorname{tors}(H_{n-1}(X,A;\mathbb{Z})) \to 0$$

   and if (X,A) is of finite type, we have:

$$0 \to \operatorname{fr}(H_n(X,A;\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \to H_n(X,A;\mathbb{Q}/\mathbb{Z}) \to \operatorname{tors}(H_{n-1}(X,A;\mathbb{Z})) \to 0$$

Hint: In this exercise you are allowed to use that $\operatorname{Tor}_{\mathbb{Z}}(\mathbb{A},\mathbb{B}) = 0$ if $\mathbb{A}$ or $\mathbb{B}$ are torsion free. The case, where $\mathbb{A}$ and $\mathbb{B}$ are finitely generated, has been studied in Exercise 9.2.



Saunders Maclane, 4 August 1909 to 14 April 2005

Exercise 10.5* (The splitting of the universal coefficient theorem is not natural.)
Consider the map $f\colon \mathbb{R}P^2 \to \mathbb{S}^2$, defined by collapsing the 1-skeleton in $\mathbb{R}P^2 = \mathbb{S}^1 \cup_\phi e^2$ to a point. Note that the attaching map $\phi\colon \mathbb{S}^1 \to \mathbb{S}^1$ of the 2-cell has degree 2 and therefore we may use Exercise 10.2. here.

1. Compute H_1 and H_2 of $\mathbb{R}P^2$ and of $\mathbb{S}^2$ with coefficients $\mathbb{Z}$ and $\mathbb{Z}/2$ directly from the cellular chain complex $C_\bullet(X)$ and $C_\bullet(X) \otimes_{\mathbb{Z}} \mathbb{Z}/2$.
2. Show that $f_*\colon H_i(\mathbb{R}P^2;\mathbb{Z}) \to H_i(\mathbb{S}^2;\mathbb{Z})$ is trivial for $i = 1$ and $i = 2$.
3. Show that $f_*\colon H_2(\mathbb{R}P^2;\mathbb{Z}/2) \to H_2(\mathbb{S}^2;\mathbb{Z}/2)$ is non-trivial.
4. Conclude that the splitting of the sequence in the universal coefficient theorem is not natural, i.e. not even for the two spaces involved here can one find splittings which commute with the homomorphisms induced by the map f. In other words: although the homology groups $H_n(X;\mathbb{Z}/2)$ are determined by $H_n(X;\mathbb{Z}/2)$ and $H_{n-1}(X;\mathbb{Z}/2)$, the induced homology morphisms $H_n(f;\mathbb{Z}/2)$ are not determined by $H_n(f;\mathbb{Z})$ and $H_{n-1}(f;\mathbb{Z})$.

Exercises

1. By taking L free with a given basis, show that the elements of $\mathrm{Tor}_1(G, C)$ can be taken to be symbols $((g_1, \ldots, g_m), x, (c_1, \ldots, c,))$ with $g_i \in G$, $c_j \in C$ and x an $m \times n$ matrix of entries from R such that $(g_1, \ldots, g_m)\, x = 0 = x\,(c_1, \ldots, c_n)'$; here the prime denotes the transpose. Describe the addition of such symbols and show that the equality of such is given by sliding matrix factors of x right and left.

2. Obtain a similar definition of Tor, (G,C).

3. Prove that Tor, $(P,C) = 0$ for $n > 0$ and P projective. (Hint: show first that it suffices to prove this when P is finitely generated.)

The exactness of (7.9) can be proved directly (i.e., without homology) as in the followipg sequence of exercises.

4. Show that the composite of two successive maps in (7.9) is zero and that the exactness of (7.9) for G finitely generated implies that for all G.

From Mac Lane: Homology, Chapter 5.7.