Aufgaben zur Topologie II

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Week 9 — Tor, Hom, Ext due by: 28.6.2017

Exercise 9.1 (Torsion in modules)
Let $\mathbb{K}$ be a commutative ring with unit. An element x in a $\mathbb{K}$-module M is called a **torsion element**, if $\lambda x = 0$ for some $0 \neq \lambda \in \mathbb{K}$. The module M is called *torsion-free* if $x = 0$ is the only torsion element; and M is called *torsion* or a *torsion module*, if all elements are torsion elements.

1. If $\mathbb{K}$ is an integral domain, then the torsion elements in M form a submodule $\mathrm{tors}(M)$ of M.
2. Find a ring $\mathbb{K}$ and a $\mathbb{K}$-module M such that the set of torsion elements is not a submodule of M.

Assume for the rest of the exercise, that $\mathbb{K}$ is an integral domain.

3. The module $M/\,\mathrm{tors}(M)$ is torsion-free.
4. If M is free, it is torsion-free.
(The converse if not true in general: take $\mathbb{K} = \mathbb{Z}$ and $M = \mathbb{Q}$, which is torsion-free but not free. However, if $\mathbb{K}$ is a principal ideal domain and M is finitely generated, then free and torsion-free are equivalent; sdee Exerc. 9.2.4 below.)
5. a) If M is torsion-free, then $\mathrm{Hom}_{\mathbb{K}}(A, M)$ is torsion-free for all A.
b) If M is torsion and finitely generated, then $\mathrm{Hom}_{\mathbb{K}}(A, M)$ is torsion for all A.
c) If A is torsion and B is torsion-free, then $\mathrm{Hom}_{\mathbb{K}}(A, B) = 0$.
6. There is a canonical isomorphism $\mathrm{tors}(A \oplus B) \cong \mathrm{tors}(A) \oplus \mathrm{tors}(B)$.
7. For any $\mathbb{K}$-homomorphism $f\colon A \to B$, $f(a)$ is torsion if a is torsion. Thus $f(\mathrm{tors}(A)) \subseteq \mathrm{tors}(B)$.
8. If $0 \to A' \to A \to A'' \to 0$ is an exact sequence, is $0 \to \mathrm{tors}(A') \to \mathrm{tors}(A) \to \mathrm{tors}(A'') \to 0$ exact?
9. What is $\mathrm{tors}(\mathbb{S}^1)$?

Exercise 9.2 (Finitely generated modules over principal ideal domains)
Let $\mathbb{K}$ be a principle ideal domain and M be any $\mathbb{K}$-module. For a scalar $\lambda \in \mathbb{K}$ we denote the multiplication $\lambda\colon M \to M$ by λ on M by the same letter and define $\lambda M = \mathrm{Im}(\lambda) = \{x \in M \mid x = \lambda y \text{ for some } y \in M\}$ and $M[\lambda] = \mathrm{Ker}(\lambda) = \{x \in M \mid \lambda x = 0\}$.

1. Show that $\lambda(M[\lambda]) = 0$ and that $(M/M[\lambda])[\lambda] = 0$.

Recall the Elementary Divisor Theorem: *A finitely generated $\mathbb{K}$-module M over a principal ideal doman $\mathbb{K}$ is a direct sum $M = \bigoplus_{i=1}^{m} \mathbb{K}/\lambda_i\mathbb{K}$ of modules $\mathbb{K}/\lambda_i\mathbb{K}$ for some $\lambda_1, \ldots, \lambda_m \in \mathbb{K}$.* [See e.g. S.Lang: Algebra, Theorem III. §7.8, p. 153]. We can exclude λ_i being a unit, but $\lambda_i = 0$ is possible.

2. Given a decomposition of M as above, describe λM and $M[\lambda]$.
3. Given a decomposition of M as above, describe $\mathrm{tors}(M)$ and $M/\,\mathrm{tors}(M)$.
4. Conclude for M finitely generated: $M/\,\mathrm{tors}(M)$ is free and $M \cong \mathrm{tors}(M) \oplus M/\,\mathrm{tors}(M)$.

Samuel Eilenberg September 30, 1913 to January 30, 1998.

Exercise 9.3 (Small computations of Hom and Ext.)
Let n and m be positive integers, and let $\gcd(m,n)$ be their greatest common divisor. Prove the following isomorphisms:

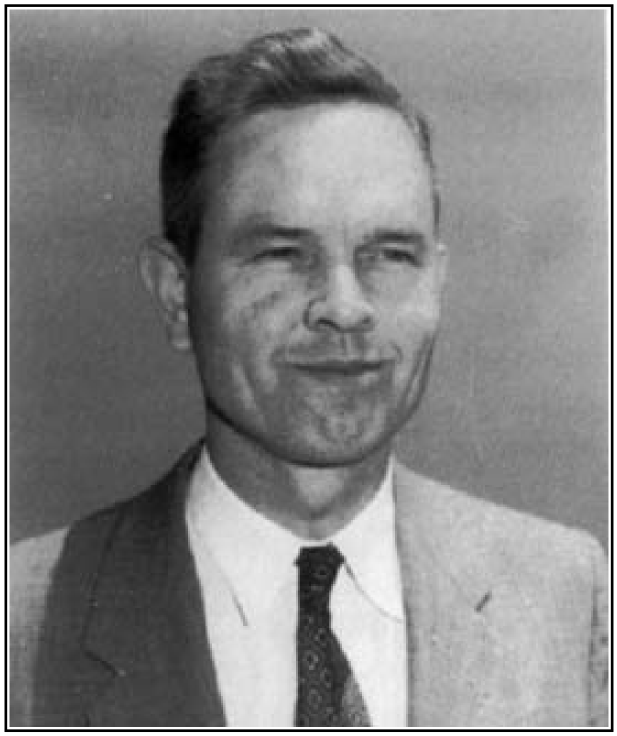
1. $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}/m) \simeq \mathbb{Z}/\gcd(m,n)$
2. $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}) \simeq 0$
3. $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/m) \simeq \mathbb{Z}/m$
4. $\mathrm{Ext}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}/m) \simeq \mathbb{Z}/\gcd(m,n)$
5. $\mathrm{Ext}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}) \simeq \mathbb{Z}/n$
6. $\mathrm{Ext}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/m) \simeq 0$

Exercise 9.4 (Limits and their cohomology)
Let S denote the limit of the direct system of 1-spheres $f_n\colon X_n = \mathbb{S}^1 \to X_{n+1} = \mathbb{S}^1$, each f_n being a map of degree 2. Compute the fundamental group, all homology groups and all cohomology groups of the space S.
What if we replace the spheres $\mathbb{S}^1$ by spheres $\mathbb{S}^k$? What if we change the degrees of the maps f_n to be powers of 2, or powers of another prime, or powers of several primes, or have all primes occur as degrees ?

Exercise 9.5* (Ext as the group of extensions.)
Let A and C be two abelian groups and consider two short exact sequences E and E' of abelian groups

$$(E) \qquad 0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\pi} C \longrightarrow 0$$

$$(E') \qquad 0 \longrightarrow A \xrightarrow{\iota'} B' \xrightarrow{\pi'} C \longrightarrow 0$$



Norman Steenrod April 22, 1910 to October 14, 1971.

Somehow against intuition, such an short exact sequence (E) is called *an extension of C by A*. We say that (E) and (E') are isomomorphic if there is an isomorphism $\phi\colon B \to B'$ such that the following diagram is commutative

$$\begin{array}{ccccccccccc} (E) & & 0 & \longrightarrow & A & \xrightarrow{\iota} & B & \xrightarrow{\pi} & C & \longrightarrow & 0 \\ & & & & \| & & \downarrow{\scriptstyle\phi} & & \| & & \\ (E') & & 0 & \longrightarrow & A & \xrightarrow{\iota'} & B' & \xrightarrow{\pi'} & C & \longrightarrow & 0 \end{array}$$

(1) Show that for $A = C = \mathbb{Z}_3$ there are two non-isomorphic short exact sequences whose middle term is isomorphic to $\mathbb{Z}/9$.

(2) Let $\mathfrak{Ext}(C; A)$ be the set of isomorphism classes of short exact sequences as above. We want to make $\mathfrak{Ext}(A; C)$ into an abelian group. Consider two classes $[E]$ and $[E']$, where E and E' are two short exact sequences as above. Consider the short exact sequence

$$(E \oplus E') \qquad 0 \longrightarrow A \oplus A \xrightarrow{\iota \oplus \iota'} B \oplus B' \xrightarrow{\pi \oplus \pi'} C \oplus C \longrightarrow 0.$$

Consider further the maps $\nabla_A\colon A \oplus A \to A$, $(a_1, a_2) \mapsto a_1 + a_2$ and $\Delta_C\colon C \to C \oplus C$, $c \mapsto (c, c)$. Show that $(\pi \oplus \pi')^{-1}(\mathrm{Im}(\Delta_C)$ contains $(\iota \oplus \iota')(\mathrm{Ker}\, \nabla_A)$ (both are subgroups of $B \oplus B'$). Show that there is a short exact sequence

$$(E'') \qquad 0 \longrightarrow A \longrightarrow (\pi \oplus \pi')^{-1}(\mathrm{Im}(\Delta_C)/(\iota \oplus \iota')(\mathrm{Ker}\, \nabla_A) \longrightarrow C \longrightarrow 0$$

We define $[E] + [E']\colon = [E'']$.

(3) Show that $\mathfrak{Ext}(C; A)$ is an abelian group.

Hint: the neutral element is the class of the split exact sequence $0 \to A \to A \oplus C \to C \to 0$; the negative of the class of $[E]$, with the notation above, is the class of the short exact sequence

$$(-E) \qquad 0 \longrightarrow A \xrightarrow{-\iota} B \xrightarrow{\pi} C \longrightarrow 0$$

(4)* If $\phi\colon A \to A'$ and $\psi\colon C' \to C$ are maps of abelian groups, construct a map of abelian groups

$$\mathfrak{Ext}(\psi;\phi)\colon \mathfrak{Ext}(C;A) \to \mathfrak{Ext}(C';A')$$

using pushouts and pullbacks.

Remark: the two bifunctors $\mathrm{Ext}(C;A)$ *and* $\mathfrak{Ext}(C;A)$ *are isomorphic as bifunctors. See for example Charles Weibel, An introduction to Homological Algebra.*