Aufgaben zur Topologie II

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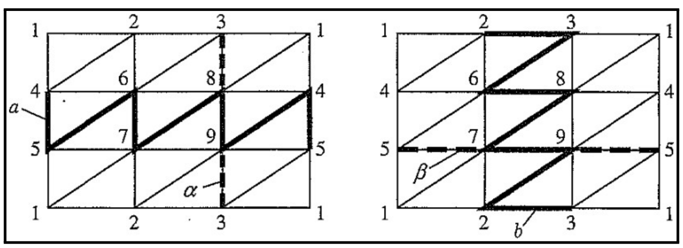
Week 8 — The uniqueness theorem, $H^1(X;\mathbb{M})$ and Bockstein homomorphisms due by: 21.6.2017

Exercise 8.1 (Simplicial cohomology of three examples)
Consider the triangulation of the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ as it is seen twice in the picture below. By (i, j), we denote the oriented edge from i to j, regarded as a simplicial 1-chain. Our ground ring here is $\mathbb{Z}$. Likewise, let $(i, j)^*$ denote the dual element, namely a 1-cochain. (*In all figures below chains and cochains are not distinguished; chains are drawn dashed, cochains are drawn thick.*)

(1) Show that the 1-chains $\alpha = (3, 9) + (9, 8) + (8, 3)$ and $\beta = (5, 7) + (7, 9) + (9, 5)$ are cycles in the simplicial chain complex obtained form the triangulation.

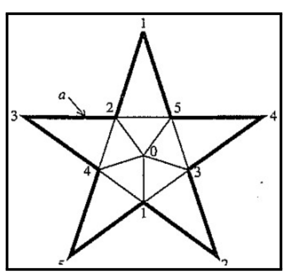
(2) Show that the 1-cochains a and b in the figure are cocycles such that $a(\alpha) = 1 = b(\beta)$ and $b(\alpha) = 0 = a(\beta)$.

(3) Use exercise 6.4 to show that a and b generate the first integral cohomology of the torus.



A triangulation of the torus. From Hausmann: Mod Two Homology and Cohomology, page 27.

Now, for the real projective plane and the Klein bottle, let the ground ring be $\mathbb{Z}/2$. One can compute the first homology and cohomology with coefficients in $\mathbb{Z}/2$ similarly (using the triangulation suggested by the two figures below). For the projective plane, the cycle α and the cocycle a satisfy $a = \alpha^*$, so one can not distinguish them in the figure.



A triangulation of the real projective plane; loc. cit., page 26.

Exercise 8.2 (Paths and integration)
A singular 1-cochain $\alpha \in S^1(X;\mathbb{G}) = \mathrm{Hom}_{\mathbb{K}}(S_1(X);\mathbb{G}) = \mathrm{Funct}(\mathfrak{B}_1(X);\mathbb{G})$ is a function on the basis of $S_1(X)$, the set $\mathfrak{B}_1(X)$ of paths $c\colon \Delta^1 \to X$, taking values in a $\mathbb{K}$-module $\mathbb{G}$. Assume α is a cocycle and show:

(1) $\alpha(c) = 0$, if c is a constant path.

(2) $\alpha(c) = \alpha(c')$, if $c \simeq c'$ relative endpoints.

(3) $\alpha(c \star c') = \alpha(c) + \alpha(c')$, if $c \star c'$ denotes the concatenation of two paths with $c(1) = c'(0)$.

(4) α is a coboundary if and only if α depends only on the endpoints (i.e. if $c(0) = c'(0)$ and $c(1) = c'(1)$ implies $\alpha(c) = \alpha(c')$).

— ***Important interlude***: *Let $X \subset \mathbb{C}$ be open and connected, $\mathbb{K} = \mathbb{G} = \mathbb{C}$ and let $f\colon X \to \mathbb{C}$ be a holomorphic function. Define*

$$I_f(c) := \int_c f(\zeta)d\zeta = \int_0^1 f(c(t))\dot{c}(t)dt$$

for a path $c \in \mathfrak{B}_1(X)$ in X. By Cauchy's theorem, I_f is a cocycle; and it has the properties (1) – (3). Moreover, we have $I_{f+g} = I_f + I_g$ and $I_{\lambda f} = \lambda I_f$ for two holomorphic functions f and g an X and a scalar $\lambda \in \mathbb{C}$.
If the integral I_f depends only on the endpoints, then

$$F(z) := I_f(c) = \int_c f(\zeta)d\zeta$$

for some path c from a fixed $z_0 = c(0)$ to $z = c(1)$ defines an indetermined integral, i.e. a (holomorphic) function $F\colon X \to \mathbb{C}$ with derivative $F' = f$ and normalized by $F(z_0) = 0$. For us, the last equation says: F is a 0-cochain with coboundary $\delta(F) = I_f$ and normalized by $F(z_0) = 0$ on the 0-chain z_0. — ***End of interlude.***

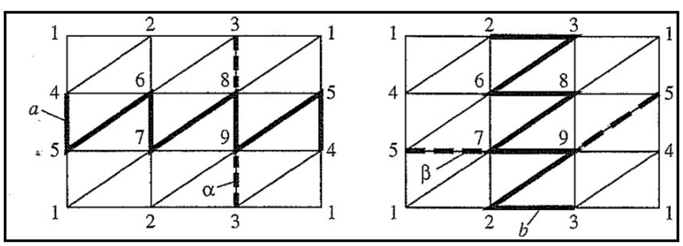
(5) Any homomorphism $\varphi\colon \pi_1(X, *) \to \mathbb{G}$ determines a cocycle α_φ by

$$\alpha_\phi(c) := \varphi[\omega_{x_0} \star c \star \overline{\omega}_{x_1}]$$

for a $c \in \mathfrak{B}_1(X)$, where we use like in Exercise 7.2(5) a system of paths ω_x from the basepoint $*$ to x, and $x_0 = c(0), x_1 = c(1)$. Vice versa, any cocycle $\alpha\colon \mathfrak{B}_1(X) \to \mathbb{G}$ determines by restriction to the closed paths c with $c(0) = c(1) = *$ a homomorphism $\varphi\colon \pi_1(X, *) \to \mathbb{G}$ by

$$\varphi([\alpha]) := \alpha(c).$$

Is this correspondence a bijection ?



A triangulation of the Klein bottle; loc. cit., page 29.

Exercise 8.3 (Uniqueness of singular homology and cohomology)
Recall that a homology theory h_* is defined to be a collection of covariant functors $(X, A) \to h_n(X, A)$, $n \in \mathbb{Z}$ from some subcategory T of topological pairs (X, A) to the category of abelian groups $\mathbb{Z} - MOD$, together with a collection of natural connecting homomorphisms $\partial^{(X,A)}\colon h_n(X, A) \to h_{n-1}(A, \emptyset)$, satisfying the Eilenberg–Steenrod axioms of homotopy invariance, of long exact homology sequences, of excision; we also assume the following *additivity axiom* (which is satisfied by all singular homology theories): If $X = \coprod_\alpha X_\alpha$ is the disjoint union of topological spaces X_α, the homomorphism $\bigoplus i^\alpha_*\colon h_n(X_\alpha, \emptyset) \to h_n(X, \emptyset)$ is an isomorphism, where $\iota^\alpha\colon X_\alpha \to X$ is the inclusion.

Assume we have two homology theories h_* and h'_* both defined on some subcategory $\mathcal{T}$ of CW pairs; suppose there is a natural transformation $\vartheta\colon h_* \to h'_*$ of homology theories, i.e., a collection $\vartheta_n\colon h_n \to h'_n$ of natural transformations commuting with the connecting homomorphisms. Using the additivity axiom for the 0-skeleton, the 5-lemma and induction along the skeleta, we proved in the lecture the following comparison theorem for homology theories:

Theorem: If $\vartheta_n\colon h_n(pt) \to h'_n(pt)$ is an isomorphism for each $n \in \mathbb{Z}$, then $\vartheta_n\colon h_n(X,A) \to h'_n(X,A)$ is an isomorphism for each CW-pair (X,A) and all $n \in \mathbb{Z}$.

In particular, if h_* and h'_* satisfy the dimension axiom (as the singular homology theories do) and $\vartheta_0\colon h_0(pt) \to h'_0(pt)$ is an isomorphism (between the coefficients), then h_* and h'_* are isomorphic theories.

Now dualize the above statement and prove:

Theorem: If $\vartheta^n\colon h^n(pt) \to h'^n(pt)$ is an isomorphism for each $n \in \mathbb{Z}$, then $\vartheta^n\colon h^n(X,A) \to h'^n(X,A)$ is an isomorphism for each CW-pair (X,A) and all $n \in \mathbb{Z}$.

Exercise 8.4 (Inclusions and quotients)
(1) Consider an m-manifolds M, a closed subset A with an open neighbourhood W. Show that $H^*(M, M \setminus A) \cong H^*(\overline{W}, \overline{W} \setminus A)$.
Example: A a submanifold of M of dimension k, W an open tubular neighbourhood of A in M; thus $\overline{W}$ is an m-manifold with boundary $\partial\overline{W}$ and $W \setminus A \simeq \partial\overline{W}$. Specifically, $H^m(M, M-p) \cong H^m(\mathbb{D}^m, \mathbb{S}^{m-1}) \cong \mathbb{Z}$ for any point $p \in M$.

Let $\iota\colon A \to X$ be the inclusion of a subspace and let $q\colon X \to (X,A)$ be the corresponding "quotient" map.

(2) If A is acyclic (i.e., $\tilde{H}^n(A) = 0$ for all $n \geq 0$), then $q^*\colon \tilde{H}^n(X,A) \to \tilde{H}^n(X)$ is an isomorphism for all $n \geq 0$.
Example: $\mathbb{R}^1 \cong A \hookrightarrow X = \mathbb{R}^3$ a knotted curve.

(3) If X is acyclic, then $\delta^*\colon \tilde{H}^{n-1}(A) \to \tilde{H}^n(X,A)$ is an isomorphism for all $n \geq 0$.
Example: $\mathbb{S}^1 \cong A \hookrightarrow X = \mathbb{R}^3$ a knot.

(4) If $\iota \simeq *$ is nullhomotopic, then for all $n \geq 0$ there is a short exact sequence

$$0 \longrightarrow \tilde{H}^{n-1}(A) \xrightarrow{\delta^*} \tilde{H}^n(X,A) \xrightarrow{q^*} \tilde{H}^n(X) \longrightarrow 0$$

Example: $A = \mathbb{S}^{n-1}$ embedded as equator into $X = \mathbb{S}^n$, with $X/A \cong \mathbb{S}^n \vee \mathbb{S}^n$; in this example the sequence splits.

Exercise 8.5* (Bockstein homomorphisms)
Any $\mathbb{K}$-linear homomorphism $\varphi\colon \mathbb{A} \to \mathbb{B}$ between two $\mathbb{K}$-modules induces a natural transformation $\varphi_\#\colon H_n(X,X';\mathbb{A}) \to H_n(X,X';\mathbb{B})$ of homology theories, namely $\phi_\#$ is induced by the chain map

$$S_\bullet(\mathrm{id}_{X,X'}) \otimes \varphi\colon S_\bullet(X,X';\mathbb{K}) \otimes \mathbb{A} \to S_\bullet(X,X';\mathbb{K}) \otimes \mathbb{B}\,.$$

(0) For any short exact sequence $\mathcal{E} : 0 \to \mathbb{A} \xrightarrow{\iota} \mathbb{B} \xrightarrow{\pi} \mathbb{G} \to 0$ of $\mathbb{K}$-modules there is a short exact sequence of chain complexes

$$0 \to S_\bullet(\mathrm{id}_{X,X'}) \otimes \mathbb{A} \xrightarrow{\mathrm{id}\otimes\iota} S_\bullet(\mathrm{id}_{X,X'}) \otimes \mathbb{B} \xrightarrow{\mathrm{id}\otimes\pi} S_\bullet(\mathrm{id}_{X,X'}) \otimes \mathbb{G} \to 0.$$

(Hint: You need that the chain complex $S_\bullet(X,X')$ consists of free $\mathbb{K}$-modules. But you do not need that the first derived functor of the tensor product vanishes, if one tensor factor is free.)

(1) Conclude that there is a long exact sequence of homology groups from different homology theories:

$$\begin{tikzcd}
\dots \arrow[r] & H_n(X,X';\mathbb{A}) \arrow[r, "\iota_\#"] & H_n(X,X';\mathbb{B}) \arrow[r, "\pi_\#"] & H_n(X,X';\mathbb{G}) \arrow[dll, "\partial^{\mathcal{E}}_\#"'] \\
& H_{n-1}(X,X';\mathbb{A}) \arrow[r, "\iota_\#"] & H_{n-1}(X,X';\mathbb{B}) \arrow[r, "\pi_\#"] & H_{n-1}(X,X';\mathbb{G}) \arrow[r] & \dots
\end{tikzcd}$$

(2) For the sequence $\mathcal{E} : 0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$ we obtain from the connecting homomorphism $\partial_{\#}^{\mathcal{E}}$ the Bockstein homomorphism

$$\beta \colon H_n(X, X'; \mathbb{Z}/2) \to H_{n-1}(X, X'; \mathbb{Z}/2)\,.$$

(3) For the sequence $\mathcal{E}' : 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$ we obtain from the connecting homomorphism $\partial_{\#}^{\mathcal{E}'}$ the Bockstein homomorphism

$$\beta' \colon H_n(X, X'; \mathbb{Z}/2) \to H_{n-1}(X, X'; \mathbb{Z})$$

with $\mathrm{Im}(\beta') = H_{n-1}(X, X'; \mathbb{Z})[2]$ and $\mathrm{Ker}(\beta') = H_n(X, X'; \mathbb{Z})/2H_n(X, X; \mathbb{Z})$.

(4) Dualize the above statements (0) to (3) to obtain the Bockstein homomorphisms in cohomology.