Aufgaben zur Topologie II

Prof. Dr. C.-F. Bödigheimer

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Week 7 — Cohomology of spheres and Hom **functor** due by: 14.6.2017

Exercise 7.1 (Generator for $H^n(\mathbb{S}^n;\mathbb{Z})$)
We consider the triangulation of $\mathbb{S}^n$ as the boundary of a $(n+1)$-simplex Δ^{n+1} and denote the simplicial chain complex by $\mathcal{C}_\bullet$, and the coboundary operator by $d\colon \mathcal{C}_k \to \mathcal{C}_{k-1}$. The dual cochain complex is denoted by $\mathcal{C}^\bullet = \mathrm{Hom}_\mathbb{Z}(\mathcal{C}_\bullet;\mathbb{Z})$, with coboundary operator $d^*\colon \mathcal{C}^{k-1} \to \mathcal{C}^k$.

(1) Determine the rank of $\mathcal{C}^k$ for all $k \geq 0$ and compute the Euler characteristic.

If $e_0, e_1, \dots, e_{n+1}$ are the vertices of Δ^{n+1}, let s_i denote the i-th face $\langle e_0, \dots, \hat{e}_i, \dots, e_{n+1}\rangle$ of Δ^{n+1}: it corresponds to a generator in $\mathcal{C}_n$, that we still denote $s_i \in \mathcal{C}_n$.
We denote by $\sigma_i = s_i^*$ the dual generator in $\mathcal{C}^n$, so

$$\sigma_i(s_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else.} \end{cases}$$

(2) Given any $i = 0, 1, \dots, n+1$ and any cochain $\alpha = \sum_{j=0}^{n+1} \lambda_j \sigma_j \in \mathcal{C}^n$ (which is necessarily a cocycle), show that there is a cochain $\beta \in \mathcal{C}^{n-1}$ such that $\alpha - d^*(\beta) = \lambda\sigma_i$ for some $\lambda \in \mathbb{Z}$.

(3) Conclude that the $\sigma_0, \sigma_1, \dots, \sigma_{n+1}$ are cohomologous to each other (with appropriate choices of signs), and each cohomology class $[\sigma_i]$ is a generator of $H^n(\mathbb{S}^n;\mathbb{Z})$.

Exercise 7.2 ($H^1(X)$ and spanning trees)
Let $\mathfrak{X}$ be a simplicial scheme, $X = |\mathfrak{X}|$ its realisation; suppose that X is connected and choose a spanning tree T of the 1-skeleton of X (which is a graph); choose a vertex v_0 as the basepoint of the fundamental group.
Each homomorphism $\varphi\colon \pi_1(X, v_0) \to \mathbb{A}$ into an abelian group $\mathbb{A}$ factors through a homomorphism $\bar{\varphi}\colon H_1(X) \to \mathbb{A}$, since $\pi_1(X, v_0)^{\mathrm{ab}} \simeq H_1(X)$.
For each φ consider the following function $\Omega = \Omega_\varphi$ defined on 1-simplices $\sigma = \{v, w\}$ with $v < w$ in the ordering of the vertices:

$$\Omega(\sigma) = \begin{cases} 0 & \text{if } \sigma \in T \\ \varphi(\omega_v \star \sigma \star \bar{\omega}_w) & \text{else,} \end{cases}$$

where ω_v is the unique path in T from v_0 to v, $\bar{\omega}_w$ is the reverse of ω_w, and σ is regarded as a path from v to w; the concatenation of paths is denoted by $\star$. (Which element of $\pi_1(X, v_0)$ would $\omega_v \star \sigma \star \bar{\omega}_w$ be if σ is a 1-simplex in T ?).

(1) Prove that Ω is a 1-cocycle in the simplicial cochain complex $\mathcal{C}^\bullet(X;\mathbb{A})$.

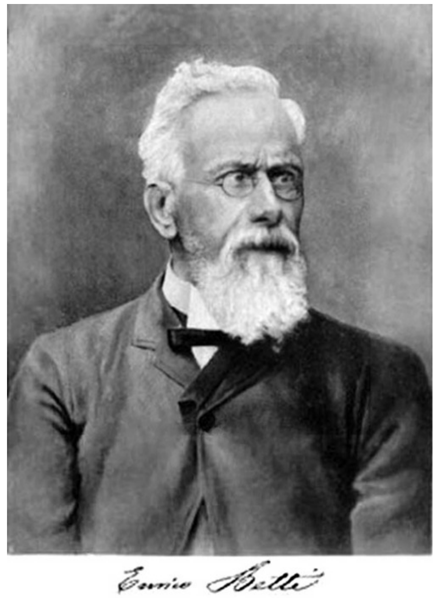
(2) Prove that Ω is not a 1-coboundary if $\bar{\varphi}$ is non-trivial.

(3) Prove that the assignment $\varphi \mapsto \Omega_\varphi$ is a homomorphism of abelian groups $\mathrm{Hom}(\pi_1(X, v_0);\mathbb{A}) \to \mathcal{C}^1(X;\mathbb{A})$. (How can $\mathrm{Hom}(\pi_1(X, v_0);\mathbb{A})$ be considered as an abelian group ?).

Remark: The homomorphism above induces a map $\mathrm{Hom}(H_1(X;\mathbb{Z});\mathbb{A}) \to H^1(X;\mathbb{A})$; it is an isomorphism and its inverse is discussed in exercise 6.3.

(4)* How does Ω_φ depend on the choice of the tree T ?

(5)* Try to construct in the same spirit a singular cocycle Ω_φ in the singular cochain complex $S^\bullet(X;\mathbb{A})$ of an arbitrary connected space X with some basepont x_0, where a homorphism $\varphi\colon \pi_1(X, x_0) \to \mathbb{A}$ is given; now use any system of paths $\omega = (\omega_x | x \in X)$ from the basepoint x_0 to x. Note that such a system is completely arbitrary and the dependence of ω_x on x need not be continuous (in the topology of X resp. of the path space of X). One knows such systems from the proof of the Seifert-van Kampen theorem.



Enrico Betti, 21 October 1823 to 11 August 1892.

Exercise 7.3 (Properties of Hom)
Let $\mathbb{K}$ be a commutative ring with unit. In what follows, we denote $\mathbb{K}$-modules by A, A_i, B, B_j and M, where $i \in I$ and $j \in J$.

1. Show that $\mathrm{Hom}_{\mathbb{K}}(A, B)$ is a $\mathbb{K}$-module.
2. Show that $\mathrm{Hom}_{\mathbb{K}}(A, \prod_{j\in J} B_j) \cong \prod_{j\in J} \mathrm{Hom}_{\mathbb{K}}(A, B_j)$.
3. Show that $\mathrm{Hom}_{\mathbb{K}}(\bigoplus_{i\in I} A_i, B) \cong \prod_{i\in I} \mathrm{Hom}_{\mathbb{K}}(A_i, B)$.
4. The evaluation map
$$\mathrm{eval}\colon \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}, M) \to M, \quad \varphi \mapsto \varphi(1),$$
is an natural isomorphism of $\mathbb{K}$-modules; in particular $\mathrm{Hom}_{\mathbb{K}}(\mathbb{K}, \mathbb{K}) \cong \mathbb{K}$.
5. Conclude from (2) − (4): if A and B are free of finite rank n resp. m, then $\mathrm{Hom}_{\mathbb{K}}(A, B)$ is free of rank mn.

Take $\mathbb{K} = \mathbb{Z}$ and let A be free with infinite basis $\mathcal{A}$; then $\mathrm{Hom}_{\mathbb{K}}(A, \mathbb{Z}) \cong \prod_{\mathcal{A}} \mathbb{Z}$. However, $\prod_{\mathcal{A}} \mathbb{Z}$ is not free (see Lam, Lectures on rings and modules, pages 22-23).

Exercise 7.4 (Hom, contravariant)
Let $\mathbb{K}$ be a commutative ring with unit and M a $\mathbb{K}$-module. Consider the functor $A \mapsto \mathrm{Hom}_{\mathbb{K}}(A, M)$.

1. For any exact sequence
$$0 \longrightarrow A_1 \xrightarrow{\iota} A_2 \xrightarrow{\pi} A_3 \longrightarrow 0$$
of $\mathbb{K}$-modules show that
$$\mathrm{Hom}_{\mathbb{K}}(A_1, M) \xleftarrow{\iota^*} \mathrm{Hom}_{\mathbb{K}}(A_2, M) \xleftarrow{\pi^*} \mathrm{Hom}_{\mathbb{K}}(A_3, M) \longleftarrow 0$$
is exact.

2. For $\mathbb{K} = \mathbb{Z}$ investigate the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

and show, that $\iota^*\colon \mathrm{Hom}_{\mathbb{K}}(\mathbb{Z}, M) \to \mathrm{Hom}_{\mathbb{K}}(\mathbb{Z}, M)$ is not surjective, if M contains elements that are not divisible by 2.

Herbert Seifert, 27 May 1907 to 1 October 1996.

Exercise 7.5* (Hom, covariant)
Let $\mathbb{K}$ be a commutative ring with unit and M a $\mathbb{K}$-module. Consider the functor $A \mapsto \mathrm{Hom}_{\mathbb{K}}(M, A)$.
Formulate and prove the covariant version of exercise 7.4: applying this functor to a short exact sequence

$$0 \longrightarrow A_1 \xrightarrow{\iota} A_2 \xrightarrow{\pi} A_3 \longrightarrow 0$$

doesn't yield in general a short exact sequence. Prove that exactness is preserved in all positions but one and find an example in which exactness in that position fails.