Aufgaben zur Topologie II

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Week 5 — Euler Characteristic again and cohomology due by: 24.5.2017

Exercise 5.1 (Euler characteristic II)
For any space X such that $H_*(X;\mathbb{Q})$ is finite dimensional in each degree and non-trivial only in finitely many degrees, let $\chi(X)$ denote its Euler charakterstic.

1. If A and B are two subcomplexes of a finite CW complex X (or of a polyhedron), such that $X = A \cup B$, then
$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B).$$
2. If A is a subcomplex of a finite CW complex X (or of a polyhedron), then
$$\chi(A) - \chi(X) + \chi(X/A) = 1.$$
Note: In case if a polyhedral pair $A \subset X$ the quotient space X/A does not have an obvious simplicial structure; but one can regard it as a CW complex.
3. Are there similar formulae for the Poincaré polynomial ?

Exercise 5.2 (Euler Characteristic III)
Let Ψ be a function, which associates to any finite CW complex X an interger $\Psi(X)$ satisfying the following axioms:

(a) For any two homeomorphic complexes X and Y we have $\Psi(X) = \Psi(Y)$.

(b) For a pair $A \subset X$ we have $\Psi(A) + \Psi(X/A) = \Psi(X) + 1$.

(c) $\Psi(\mathbb{S}^0) = 2$, where $\mathbb{S}^0 = \{\pm 1\} \subset \mathbb{R}^1$ is the sphere of dimension zero.

1. Show that $\Psi(X) = \chi(X)$ for all CW complexs X.
Hint: first show that $\Psi(B^k) = 1$ for a k-dimensional closed disk B^k, by subdividing the disk in two disks and using the second axiom; then prove the equality for spheres, and eventually for a general X.
2. Is there a similar axiomatic description of the Poincaré polynomial?

Observe that in axiom (a), instead of requiring that Ψ is invariant under homeomorphic maps we could require that Ψ is a homotopy invariant.

Exercise 5.3 (Cochain complexes)
Consider cochain complexes $A^\bullet$ and $B^\bullet$ over the ring $\mathbb{K}$. If $\phi\colon A^\bullet \to B^\bullet$ is a cochain map, let $\phi^*\colon H^*(A^\bullet) \to H^*(B^\bullet)$ denote the induced map in cohomology.

1. Show that $(\mathrm{id}_{A^\bullet})^* = \mathrm{id}_{H^*(A^\bullet)}$ and $(\psi \circ \phi)^* = \psi^* \circ \phi^*$.
2. If $\phi = 0$ is the zero map, then $\phi^* = 0$.
3. If $\phi \cong \psi$, then $\phi^* = \psi^*$.
4. For the direct sum of cochain complexes and cochain maps we have

$$H^*(A^\bullet \oplus B^\bullet) \cong H^*(A^\bullet) \oplus H^*(B^\bullet)$$

and

$$(\alpha \oplus \beta)^* = \alpha^* \oplus \beta^* \,.$$

Der zweite Abschnitt giebt die Construction der Riemann'schen Flächen, welche eine endliche Gruppe von eindeutigen Transformationen in sich besitzen, aus unabhängigen Bestimmungsstücken. Man erhält die allgemeinste derartige Fläche, wenn man eine beliebig gewählte Fläche Φ in r Exemplaren nimmt, diese Exemplare auf einander legt und in geeigneter Weise zu einer einzigen Fläche mit einander verbindet. Die so entstehenden Flächen sind keine anderen, als die von Herrn Klein eingeführten und von Herrn Dyck nach verschiedenen Richtungen untersuchten „regulären“ Riemann'schen Flächen, welche die Verzweigung von Galois'schen Resolventen darstellen*).

Ist eine Riemann'sche Fläche F vom Geschlecht p r-blättrig über der Fläche Φ vom Geschlecht π ausgebreitet, so besteht die Gleichung

$$2p - 2 = W + r(2\pi - 2),$$

wo W die Gesammtzahl der Verzweigungen von F in Bezug auf Φ bedeutet.

From Hurwitz: Über algebraische Gebilde mit eindeutigen Transformationen in sich, page 2. Hurwitz theorem — relating the Euler characteristc of a surface with the ramification indices of a branched covering — has been generalized to the Riemann–Hurwitz formula.

Exercise 5.4 (Classification of cochain maps)
Let $(A_\bullet, \partial^A)$ and $(B_\bullet, \partial^B)$ be chain complexes over a ring $\mathbb{K}$. We define a new chain complex $\mathcal{H}_\bullet = \mathcal{H}(A,B)_\bullet$ by setting

$$\mathcal{H}_n = \mathcal{H}(A,B)_n := \prod_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathbb{K}}(A_i, B_{i+n})\,.$$

So an element $f \in \mathcal{H}_n$ is a collection $(f_i)_{i\in\mathbb{Z}}$, where $f_i\colon A_i \to B_{i+n}$.
The differential $d\colon \mathcal{H}_n \to \mathcal{H}_{n-1}$ is given by

$$d\left((f_i)_{i\in\mathbb{Z}}\right) = \left(\partial^B \circ f_i - (-1)^n f_{i-1} \circ \partial^A\right)_{i\in\mathbb{Z}}\,.$$

1. Show that $d \circ d = 0$.
2. Let $(A'_\bullet, \partial^{A'})$ and $(B'_\bullet, \partial^{B'})$ be two other chain complexes over $\mathbb{K}$, and let $\alpha\colon A'_\bullet \to A_\bullet$ and $\beta\colon B_\bullet \to B'_\bullet$ be chain maps (of degree 0). Define a map $\phi = \mathcal{H}(\alpha,\beta)\colon \mathcal{H}(A,B) \to \mathcal{H}(A',B')$ by setting

$$\phi_n = \mathcal{H}(\alpha,\beta)_n = \prod_{i\in\mathbb{Z}} \mathrm{Hom}(\alpha_i, \beta_{i+n})\colon \mathcal{H}(A,B)_n \to \mathcal{H}(A',B')_n,$$

where $\mathrm{Hom}(\alpha_i, \beta_{i+n})$ maps $f_i \colon A_i \to B_{i+n}$ to $\beta_{i+n} \circ f_i \circ \alpha_i \colon A'_i \to B'_{i+n}$.

Show that ϕ is a chain map.

3. Show that $f = (f_i)_{i \in \mathbb{Z}} \in \mathcal{H}(A, B)_n$ is a cycle if and only if $f \colon A_\bullet \to B_\bullet$ is a chain map of degree n.

4. Show that a cycle $h = (h_i)_{i \in \mathbb{Z}} \in \mathcal{H}(A, B)_n$ is a boundary if and only if $h \colon A_\bullet \to B_\bullet$ is chain homotopic, as a chain map, to the zero map.
   In particular $H_0(\mathcal{H}_\bullet)$ can be identified with the group of homotopy classes of (degree 0) chain maps $A_\bullet \to B_\bullet$, where the group structure is given by (degreewise) sum of chain maps.

5. Show that an element $[f] = [(f_i)_{i \in \mathbb{Z}}] \in H_0(\mathcal{H}_\bullet)$ induces a homomorphism of graded vector spaces $\theta([f]) \colon H_*(A_\bullet) \to H_*(B_\bullet)$. Show that θ itself is a group homomorphism
$$\theta \colon H_0(\mathcal{H}(A, B)_\bullet) \to \mathrm{Hom}(H_*(A_\bullet), H_*(B_\bullet)).$$

Example 6. **Let $f(x, y)$ be a real valued function of class C^∞ (i.e., with continuous partial derivatives of all orders) defined in a connected open set D of points (x, y) in the Cartesian plane. For fixed D, the set A of all such functions is an abelian group under the operation of addition of function values. Take C to be the direct sum $A \oplus A \oplus A \oplus A$; an element of C is then a quadruple (f, g, h, k) of such functions, which we denote more suggestively as a formal "differential":**
$$(f, g, h, k) = f + g\,dx + h\,dy + k\,dx\,dy.$$
Define $d \colon C \to C$ by setting
$$d(f, g, h, k) = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \left(\frac{\partial h}{\partial x} - \frac{\partial g}{\partial y}\right) dx\,dy.$$
That $d^2 = 0$ is a consequence of the fact that $\dfrac{\partial^2 f}{\partial x\,\partial y} = \dfrac{\partial^2 f}{\partial y\,\partial x}$. Any cycle in C is a sum of the following three types: a constant $f = a$; an expression $g\,dx + h\,dy$ with $\partial g/\partial y = \partial h/\partial x$ (in other words, an exact differential); an expression $k\,dx\,dy$. If the domain D of definition is, say, the interior of the square we can write the function k as $\partial h/\partial x$ for a suitable h, while any exact differential can be expressed (by suitable integration) as the differential of a function f. Hence, for this D the only homology classes are those yielded by the constant functions, and $H(C)$ is the additive group of real numbers. The same conclusion holds if D is the interior of a circle, but fails if D is, say, the interior of a circle with the origin deleted. In this latter case an exact differential need not be the differential of a function f. For example $(-y\,dx + x\,dy)/(x^2 + y^2)$ is not such.

From Mac Lane: Homology, pages 38 and 39.

Exercise 5.5* (De Rham cohomology of a manifold)
Let M denote a smooth m-dimensional manifold, $\tau \colon T(M) \to M$ its tangent bundle and $\tau^* \colon T^*(M) \to M$ its cotangent bundle (the dual of τ). We denote by $\Lambda^k(T^*(M)) \to M$ the k-th exterior power of the cotangent bundle with fibers the k-th exterior powers of $T^*_x(M)$, the tangent space at $x \in M$; so $\Lambda^0(T^*(M))$ is the 1-dimensional trivial bundle, $\Lambda^1(T^*(M))$ is the cotangent bundle itself, and $\Lambda^m(T * (M))$ is a 1-dimensional bundle (and trivial if and only if M is orientable). A k-form on M is a section of $\Lambda^k(T^*(M))$; thus it is a function for $k = 0$, a covector field for $k = 1$, and for orientable M a volume form for $k = m$. We denote the real vector space of k-forms by $\Omega^k(M)$. The exterior derivative of forms gives a linear map
$$d \colon \Omega^k(M) \to \Omega^{k+1}(M).$$

1. Give the differential d in local coordinates.

2. Show that d is a differential, i.e., $d \circ d = 0$.

3. Show that a smooth map $f\colon M \to N$ induces a map of cochain complexes $\Omega(f)\colon \Omega^*(N) \to \Omega^*(M)$ with $\Omega(g \circ f) = \Omega(f) \circ \Omega(g)$ and $\Omega(\mathrm{id}_M) = \mathrm{id}_{\Omega^*(M)}$.

This cochain complex is called the *de Rham complex* and its cohomology is called the *de Rham cohomology* $H^*_{\mathrm{dR}}(M)$ of the manifold M; later we will see that it is isomorphic to the singular cohomology $H^*(M; \mathbb{R})$ with real coefficients.