Aufgaben zur Topologie II

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Sommersemester 2017

Week 4 — Simplicial approximations and Lefschetz fixed point theorem due by: 17.05.2017

Exercise 4.1 (Relative simplicial approximation)
Let $\mathfrak{X}$ be a finite simplicial scheme, let $\mathfrak{X}' \subseteq \mathfrak{X}$ be a subscheme (what does this mean ?), and let $\mathfrak{Y}$ be another finite simplicial scheme. Let $f\colon |\mathfrak{X}| = X \to |\mathfrak{Y}| = Y$ be a continuous map, such that its restriction on $X' = |\mathfrak{X}'| \subset X = |\mathfrak{X}|$ is simplicial: this means that there is a simplicial map $\varphi'\colon \mathfrak{X}' \to \mathfrak{Y}$ such that $|\varphi'| = f|_{X'}$.

(1) Define the *relative barycentric subdivision* $\mathrm{BSD}(\mathfrak{X}, \mathfrak{X}')$ as follows: the new vertex set $\mathrm{BSD}(\mathfrak{X}, \mathfrak{X}')_0$ consits of the old vertex set $\mathfrak{X}'_0$ and an additional vertex for each simplex of $\mathfrak{X}$ which is not contained in $\mathfrak{X}'$ — thought of as barycenter; simplices of $\mathrm{BSD}(\mathfrak{X}, \mathfrak{X}')$ have the form

$$\Sigma = \left\{v'_0, \ldots, v'_p, \sigma_{p+1}, \ldots, \sigma_n\right\}, \quad -1 \leq p \leq n,$$

where there may be no v'_i's (if $p = -1$) or no σ_j's (if $p = n$) and where

- $\tau = \left\{v'_0, \ldots, v'_p\right\}$ is a simplex in $\mathfrak{X}'$, if non-empty;
- $\tau \subset \sigma_{p+1} \subset \cdots \subset \sigma_n$ is a flag of ascending simplices in $\mathfrak{X}$. There is an obvious total ordering of these vertices.

Show that $\mathrm{BSD}(\mathfrak{X}, \mathfrak{X}')$ is a simplicial scheme that contains $\mathfrak{X}'$ as subscheme.

(2) Show that the geometric realisation of $\mathrm{BSD}(\mathfrak{X})$ is canonically homeomorphic to X, through a homeomorphism that restricts on X' to the inclusion $X' \subset X$.

(3) Define recursively $\mathrm{BSD}^{k+1}(\mathfrak{X}, \mathfrak{X}') := \mathrm{BSD}(\mathrm{BSD}^k(\mathfrak{X}, \mathfrak{X}'), \mathfrak{X}')$. Show that if k is large enough, there is a simplicial map $\varphi\colon \mathrm{BSD}^k(\mathfrak{X}, \mathfrak{X}') \longrightarrow \mathfrak{Y}$ such that

- the restriction of φ to the subscheme $\mathfrak{X}'$ is the given map φ' and
- the map $|\varphi|\colon |\mathrm{BSD}^k(\mathfrak{X}, \mathfrak{X}')| \to Y$ is homotopic to f relative to X'.
  (Here f is seen as a map $|\mathrm{BSD}^k(\mathfrak{X}, \mathfrak{X}')| \to Y$ under the canonical homeomorphism $|\mathrm{BSD}^k(\mathfrak{X}, \mathfrak{X}')| \cong X$.)

Exercise 4.2 (Fibers of simplicial maps)
Let $\varphi\colon \mathfrak{X} \to \mathfrak{Y}$ be a simplicial map between simplicial schemes.

(1) Let $\mathfrak{Y}'$ be a subcomplex of $\mathfrak{Y}$. Show that $\mathfrak{X}' := \varphi^{-1}(\mathfrak{Y}') \subseteq \mathfrak{X}$ is a subcomplex of $\mathfrak{X}$.

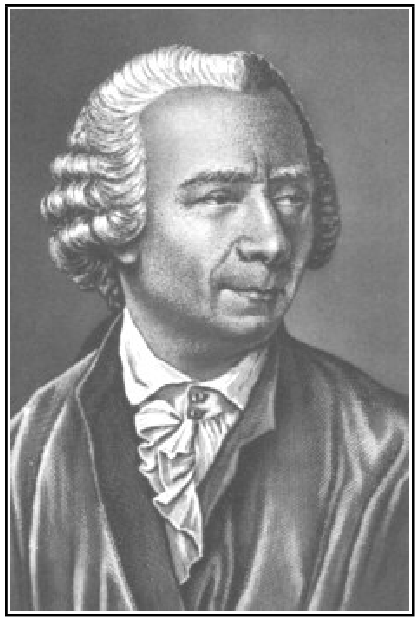
We now consider the special case where $\mathfrak{Y}' = \langle\tau\rangle$ is *generated* by a simplex $\tau \in \mathfrak{Y}$, that is, $\mathfrak{Y}' = \{\tau' \in \mathfrak{Y} | \tau' \subseteq \tau\}$. Clearly, $|\mathfrak{Y}'| = \Delta(\tau) \subseteq |\mathfrak{Y}|$. We denote by

$$f = |\varphi|\colon X = |\mathfrak{X}| \to Y = |\mathfrak{Y}|$$

the induced map between the geometric realisations. Let $\sigma \in \varphi^{-1}(\tau)$.

(2) Show that $\dim(\sigma) \geq \dim(\tau)$.

(3) Let y be a point in $|\mathfrak{Y}|$ that lies in $\Delta(\tau)$ but not on its boundary. Prove that $f^{-1}(y) \cap \Delta(\sigma) \subseteq |\mathfrak{X}|$ is homeomorphic to a product of simplices with total dimension k, where $k = \dim(\sigma) - \dim(\tau)$.



Leonhard Euler; 15. April 1707 in Basel bis 18. September 1783 in Sankt Petersburg

Exercise 4.3 (Poincaré polynomial)
Let $V = \oplus_{k\geq 0} V_k$ be a graded (and bounded below) vector space over a field $\mathbb{F}$. We call

$$P_V(t) := \sum_{i=0}^{\infty} \dim(V_i) t^i$$

its Poincaré polynomial (although it is a formal power series).

1. Compute $P_{\mathbb{F}[X]}$ for $\mathbb{F}[X]$, the polynomial ring in the indeterminant X of degree n.
2. Compute $P_{\Lambda_{\mathbb{F}}[X]}$ for $\Lambda_{\mathbb{F}}[X]$, the exteriour algebra in the indeterminant X of degree n.
3. Show $P_{V\oplus W}(t) = P_V(t) + P_W(t)$.
4. Show $P_{V\otimes W}(t) = P_V(t) \cdot P_W(t)$.
5. Show $P_V(t) = P_U(t) + P_W(t)$, if there is a short exact sequence $0 \to U \to V \to W \to 0$ of graded vector spaces, i.e., for each degree n we have a short exact sequence $0 \to U_n \to V_n \to W_n \to 0$ of vector spaces.
6. Let $(C_\bullet, d)$ a chain complex of vector spaces of finite type, over a field. Forgetting the differential d, we can consider $C_\bullet$ as a graded vector space; and similarly the cycles $Z_n := \ker(d)$, the boundaries $B_n := \mathrm{im}(d)$ and the homology $H_n := H_n(C_\bullet)$. All are of finite type. Using this, prove the equality

$$P_C(t) - P_H(t) = (1+t)P_B(t),$$

   and thus $P_C(-1) = P_H(-1)$.

These formulas and in particular the statement (6) (and the way it is proved) should remind you of the Euler characteristic of graded vector spaces (and chain complexes); no wonder, — because the Poincare polynomial is a generalization of the Euler characteristic:

$$\chi(C_\bullet) = \sum_i (-1)^i \dim(C_i) = P_C(-1).$$

Exercise 4.4 (Simplicial homotopy)
Let $f, g\colon \mathfrak{X} \to \mathfrak{Y}$ be simplicial maps, and suppose that the corresponding maps $|f|, |g|\colon X = |\mathfrak{X}| \to Y = |\mathfrak{Y}|$ are homotopic as continuous maps (so the maps F_t of a homotopy with $f = F_0$ and $g = F_1$ need not be simplicial for all $0 < t < 1$. Recall the simplicial structure on $X \times I$ from exercise 3.1, where I, the unit interval, is given the simplicial structure of the standard 1-simplex Δ^1.

(1) Is it always true that there is a simplicial map $X \times I \to Y$ restricting to $|f|$ on $X \times \{0\}$ and to $|g|$ on $X \times \{1\}$? What if we consider on I the simplicial structure with k 1-simplices, and we still consider on $X \times I$ the simplicial structure given by exercise 3.1?
Hint: Consider $X = \partial\Delta^2 \cong \mathbb{S}^1$ and Y is the surface of the icosahedron with two opposite faces removed. Let f and g be simplicial homeomorphisms of X into the two boundary components of Y, such that f and g are homotopic as maps $X \to Y$.

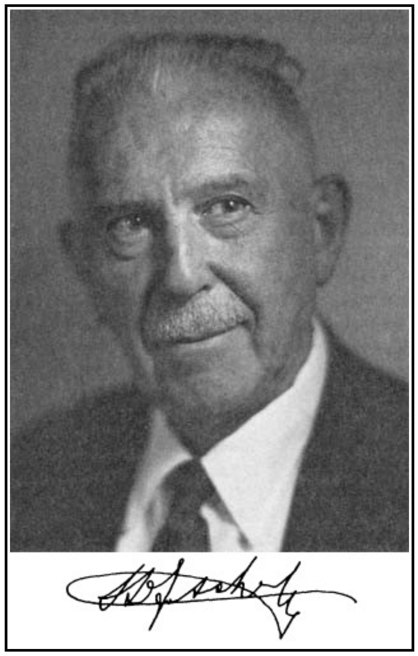
(2) Apply exercise 4.1 to show that there is a suitable simplicial subdivision of $X \times I$ and a simplicial map $\Phi\colon X \times I \to Y$ extending f and g on $X \times \{0\}$ and $X \times \{1\}$ respectively. This is called a *simplicial homotopy*.

Marius Sophus Lie; 17. Dezember 1842 in Nordfjordeid bis 18. Februar 1899 in Kristiania.

Exercise 4.5* (Euler, Lie and Lefschetz)
Let $G = |\mathfrak{G}|$ be a compact connected Lie group of positive dimension, i.e., $\mathfrak{G}$ is a finite simplicial scheme and its realization $|\mathfrak{G}|$ is homeomorphic to a Lie group G. (It is actually a theorem that every compact Lie group G is homeomorphic to the geometric realisation of some finite simplicial scheme).

(1) Show that $\chi(G) = 0$.
Hint: Consider the map $f_g : x \mapsto g \cdot x$ for some fixed $g \in G$. It is homotopic to the identity of G (why ?), but it has no fixed points (why ?).

(2) Deduce that a sphere of even dimension cannot carry the structure of a Lie group. (Actually the only spheres admitting a Lie group structure are $\mathbb{S}^1 = \mathrm{SO}(2)$ and $\mathbb{S}^3 = \mathrm{SU}(2)$. It is a hard theorem for which K-theory is needed.)



Solomon Lefschetz; 3. September 1884 in Moskau bis 5. Oktober 1972 in Princeton, New Jersey, USA.