Aufgaben zur Topologie II

Prof. Dr. C.-F. Bödigheimer

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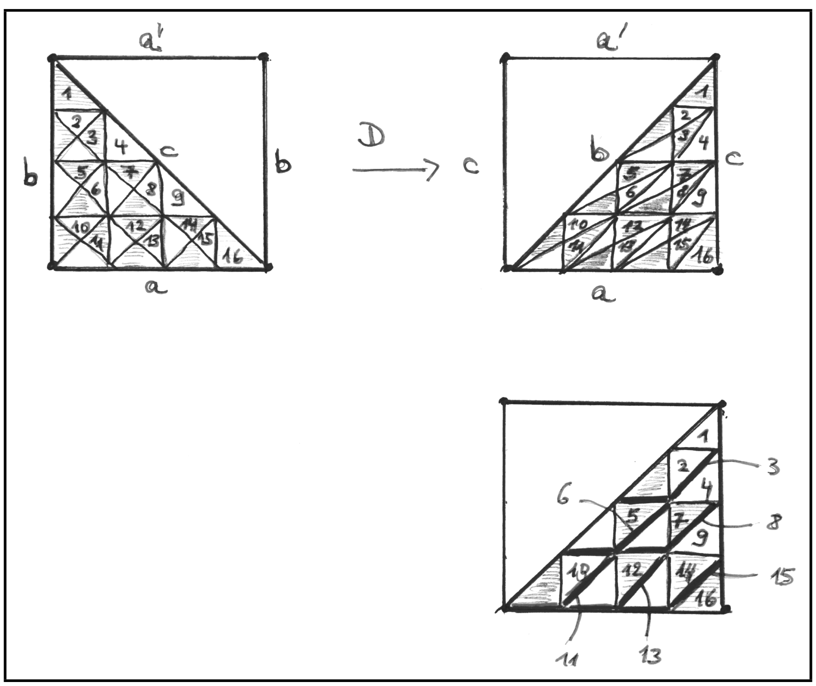
Week 3 — Simplicial maps, triangulated manifolds and homology due by: 10.05.2017

Exercise 3.1 (Triangulated manifolds)
If $\mathfrak{X}$ is a finite simplicial scheme such that $X = |\mathfrak{X}|$ is a manifold (with or without boundary), we call $\mathfrak{X}$ a triangulation of the manifold X. We want a criterion to recognize whether $|\mathfrak{X}|$ is a manifold or not.

Let σ be a k-simplex and let $\tau_1, \tau_2, \ldots$ be all simplices containing σ properly. Then $\sigma_i := \tau_i - \sigma$ is the subsimplex of τ_i 'opposite' σ. The *star* of σ, denoted by $\mathrm{Star}(\sigma)$, is the union of the interior of $\Delta(\sigma)$ and all the interiors of $\Delta(\tau_i)$. The *link* of σ, denoted by $\mathrm{Link}(\sigma)$, is the union of all $\Delta(\sigma_i)$. (Note that the link of a simplex is a subcomlex, in contrast to the star.) Consider the following link conditions:

$$L(k): \text{The link } \mathrm{Link}(\sigma) \text{ of any } k\text{-simplex } \sigma \text{ is homeomorphic to a sphere of dimension } n-k-1.$$

We remark (but will not use it here) that $L(0)$ implies all the $L(k)$ for higher k; see quotation from Thurston's book.
Assume $\mathfrak{X}$ is finite of dimension n and satisfies the conditions $L(k)$ for all $k = 0, \ldots, n$. Show that $|\mathfrak{X}|$ is a manifold of dimension n without boundary. (Hint: Consider first the case of a vertex $\sigma = \{v\}$ and study the cones of the σ_i with cone point v. Then consider the general case, namely a point x in the interior of a simplex σ and study the joins of the σ_i with σ.



A Dehn twist D on a tube, realized as a shear map on a square (with right and left side identified). The right lower picture shows D after a homotopy to simplicial map; the thick edges are the images of a the triangles as named.

Exercise 3.2 (Barycentric subdivision)
Given a simplicial scheme $\mathfrak{X}$ we denote by $\mathrm{BSD}(\mathfrak{X})$ its barycentric subdivision. We know the map

$$b\colon |\mathrm{BSD}(\mathfrak{X})| \to |\mathfrak{X}|,$$

which is given as follows. A vertex in $\mathrm{BSD}(\mathfrak{X})$, i.e., a simplex $\sigma = \{v_0, \ldots, v_n\}$ in $\mathfrak{X}$, is sent to its barycenter

$$b(\sigma) := \left[\sigma; \frac{1}{n+1}, \ldots, \frac{1}{n+1}\right] = \frac{1}{n+1}\, v_0 + \ldots + \frac{1}{n+1}\, v_n$$

in $\Delta(\sigma)$. (N.B.: we use the same notation b for the map to be defined as we used for the barycenter.) A l-simplex in $\mathrm{BSD}(\mathfrak{X})$, i.e. a flag $\Phi = (\sigma_0 \subset \sigma_1 \subset \ldots \subset \sigma_l)$ of length $l+1$ of simplices in $\mathfrak{X}$, is sent to

$$b([\Phi; s_0, \ldots, s_l]) := s_0\, b(\sigma_0) + s_1\, b(\sigma_1) + \ldots + s_l\, b(\sigma_l)\,,$$

where the sum on the right-hand side is to be understood as an 'affine combination' in the maximal $\Delta(\sigma_l)$, which contains all $\Delta(\sigma_k)$ and thus all barycenters for $k = 0, \ldots, l$. (Note that b is not simplicial.)

1. Show that b is a homeomorphism.
2. We define a map $\beta\colon \mathrm{BSD}(\mathfrak{X}) \to \mathfrak{X}$ by giving it on vertices of $\mathrm{BSD}(\mathfrak{X})$, i.e., on simplices $\sigma = \{v_{i_0} < v_{i_1} < \ldots < v_{i_n}\}$ of $\mathfrak{X}$, setting $\beta(\sigma) = \max(\sigma) = v_{i_n}$, the maximal vertex of σ in the total ordering of the vertices $\mathfrak{X}_0$. Show that β is a simplicial map.
3. Show that $|\beta|$ is homotopic to b. (Hint: See the figure below.)

Exercise 3.3 (Orientation and orientation class)
Let $\mathfrak{X}$ be a triangulation of a compact n-manifold without boundary, i.e., $\mathfrak{X}$ is a finite simplicial scheme and $M = |\mathfrak{X}|$ is a compact manifold of dimension n without boundary. An orientation on $\mathfrak{X}$ is a function $\mathcal{O}\colon \mathfrak{X}_n \to \{\pm 1\}$ on the n-simplices such that

$$(-1)^i \mathcal{O}(\sigma) = -(-1)^j \mathcal{O}(\tau) \quad \text{for any two } n\text{-simplices } \sigma \text{ and } \tau \text{ with a common face } \partial_i(\sigma) = \partial_j(\tau).$$

We call $\mathfrak{X}$ *orientable*, if it admits an orientation.

1. If $\mathcal{O}$ is an orientation, then $-\mathcal{O}$ is also an orientation (called the *opposite orientation*).
2. Assume $\mathfrak{X}$ is orientable and connected (which means $|\mathfrak{X}|$ is connected). Show that $\mathfrak{X}$ has exactly two orientations. (Hint: Consider for two orientations $\mathcal{O}_1, \mathcal{O}_2$ the function $\lambda(\sigma) := \mathcal{O}_1(\sigma)\mathcal{O}_2(\sigma)$ and show that it is constant, if $\mathfrak{X}$ is connected.)
3. Assume $\mathfrak{X}$ is orientable and connected; recall that $\mathfrak{X}$ is finite. Consider the simplicial n-chain $U = \sum_\sigma \mathcal{O}(\sigma)\, \sigma$ in the simplicial chain complex $\mathcal{C}_\bullet(\mathfrak{X})$, the sum running over all n-simplices. Show that $U = U_{\mathcal{O}}$ is a cycle.
4. Since $U_{\mathcal{O}}$ can not be a boundary (why ?), it represents a non-trivial homology class $[U_{\mathcal{O}}]$ in $H_n(\mathfrak{X})$, called the *orientation class*. Check the equality $[U_{-\mathcal{O}}] = -[U_{\mathcal{O}}]$.
5. Consider the singular chain $V = \sum_\sigma \mathcal{O}(\sigma)\, f_\sigma$ in the singular chain complex $S_\bullet(M)$, where $f_\sigma\colon \Delta^n \to M$ is the characteristic map $f_\sigma(t_0, \ldots, t_n) = [\sigma; t_0, \ldots, t_n]$ of the simplex $\Delta(\sigma)$. Show that $V = V_{\mathcal{O}}$ is a cycle.
6. Prove that under the isomorphism $H_n(\mathfrak{X}) := H_n(\mathcal{C}_\bullet(\mathfrak{X})) \longrightarrow H_n(S_\bullet(|\mathfrak{X}|)) =: H_n(|\mathfrak{X}|)$ the homology classes $[U_{\mathcal{O}}]$ and $[V_{\mathcal{O}}]$ correspond to each other. In particular $V = V_{\mathcal{O}}$ is not a boundary (this is not at all obvious in the chain complex $S_\bullet(|\mathfrak{X}|)$).

Proposition 3.2.5 (spherical links imply manifold). *Let X be a triangulated space of dimension n. If the link of every simplex of dimension p is homeomorphic to an $(n-p-1)$-sphere, X is a topological manifold.*

In fact, the conclusion is true whenever the links of vertices are $(n-1)$-spheres.

***Proof of* 3.2.5. By Exercise 3.2.4, every point in X has a neighborhood of the form $D^p \times CS^{n-p-1}$, which is homeomorphic to $D^p \times D^{d-p}$, since the cone on the sphere is a ball. These neighborhoods cover X, so X is a manifold.** $\boxed{3.2.5}$

Exercise 3.2.6. Show that a manifold obtained by gluing is orientable if and only if, when the faces of each simplex are oriented consistently, all face identifications are orientation-reversing (cf. Exercise 1.3.2).

It would be natural to suppose that the converse of Proposition 3.2.5 is true as well, but this is not the case if $n \geq 5$. The first counterexamples were found by R. D. Edwards. Example 3.2.11 below gives a somewhat different counterexample, found by Cannon [Can79].

(The right criterion is this: the polyhedron of a simplicial complex is a topological manifold if and only if the link of each cell has the homology of a sphere, and the link of every vertex is simply connected. The proof is far beyond the scope of the current discussion.)

In three dimensions this sort of thing can't happen:

Proposition 3.2.7 (manifolds have spherical links in dimension three). *A three-dimensional gluing is a three-manifold if and only if the link of every vertex is homeomorphic to S^2.*

A quotation from Thurston's book, pages 120, 121.

Exercise 3.4 (n-dimensional gluings)
An n-dimensional gluing $\mathfrak{G}$ is a collection D_α of n-simplices $D_\alpha = \Delta^n_\alpha$, $\alpha \in A$, together with a complete matching μ of the faces, i.e. of the index pairs (α, i) for $i = 0, \ldots, n$ and $\alpha \in A$. (No index pair is matched with itself, and each index pair is matched.) The geometric realization $|\mathfrak{G}|$ is the space

$$|\mathfrak{G}| = \left(\bigsqcup_\alpha D_\alpha \right) / \sim \, ,$$

where $(D_\alpha; t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_n) \sim (D_\beta; t_0, \ldots, t_{j-1}, 0, t_j, \ldots, t_n)$ if (α, i) and (β, j) are matched. Here we write $(D_\alpha; t_0, \ldots, t_n)$ for an arbitrary point in $D_\alpha = \Delta^n_\alpha$.

(1) Note that the geometric realization is equipped with the structure of a CW complex of dimension n with the D_α as n-cells. Can you find a simplicial scheme $\mathfrak{X}$ of dimension n with the D_α as n-simplices together with a canonical homemomorphism $|\mathfrak{X}| \cong |\mathfrak{G}|$? What can go wrong?

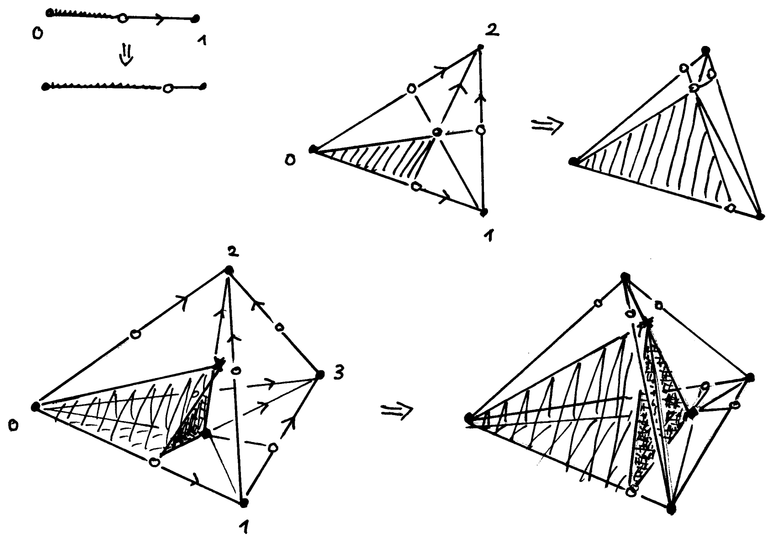
For simplicity, let us assume there is a simplicial scheme $\mathfrak{X}$ of dimension n with the D_α as n-simplices together with a canonical homemomorphism $|\mathfrak{X}| \cong |\mathfrak{G}|$.

(2) Check that the link condition $L(n-1)$ is satisfied. However the condition $L(k)$ may not be satisfied for $k < n-1$.

Thus $\mathfrak{G}$ may not be a manifold. Nevertheless, we define an orientation $\mathcal{O}$ to be a function on the n-simplices $D_\alpha \mapsto \mathcal{O}(D_\alpha) = \pm 1$ such that $(-1)^i \mathcal{O}(D_\alpha) = -(-1)^j \mathcal{O}(D_\beta)$ if $\partial_i(D_\alpha)) = \partial_j(D_\beta)$, i.e. (α, i) and (β, j) are matched.

(3) If $\mathfrak{G}$ is orientable and connected, then $H_n(|\mathfrak{G}|) \cong \mathbb{Z}$. Show that the chain $C = \sum_\alpha \mathcal{O}(D_\alpha)\, D_\alpha$ is a cycle in the

simplicial cell complex, which represents a generator. We call it the fundamental class $u_{\mathfrak{G}} = [C] \in H_n(|\mathfrak{G}|)$.



A homotopy between $|\beta|$ and b for simplices of dimensions 1,2,3. The arrows indicate the movement of the barycenters during the homotopy; all old vertices (the vertices of $\mathfrak{X}$) are fixed. There is one flag $\Phi = (\{v_0\}, \{v_0, v_1\}, \ldots, \{v_0, \ldots, v_n\})$, a subdivision simplex, whose image under $|\beta|$ covers the entire old simplex $\Delta(\sigma)$; all other subdivision simplices are mapped to faces of $\Delta(\sigma)$.

Exercise 3.5* (Gluings and singular homology classes)

Now let X be an arbitrary space and let $C = \sum_{i=1}^{r} n_i s_i$ be a singular n-chain in X, so a finite sum with $n_i \in \mathbb{Z}$ and $s_i \colon \Delta^n \to X$ continous maps. We can associate a gluing by first taking $l = |n_1| + \ldots + |n_r|$ copies of Δ^n, indexed by $\alpha = 1, \ldots, l$. If C is a cycle, then $\partial(C) = 0$; thus there must be (at least one) complete matching of all the faces of the copies of Δ^n, such that two matched faces are mapped *in the same way* to X, and their contributions to $\partial(C)$ cancel each other. Notice that in particular the number $(n+1)l$ must be even. We choose such a matching μ and call the resulting gluing $\mathfrak{G}$.

(1) Show that the assignment $\mathcal{O}(D_\alpha) = +1$ gives an orientation, that we call the *constant* orientation.

Let $P(\mathfrak{G}) = |\mathfrak{G}|$ denote the geometric realization of this gluing.

(2) Define a continuous map $f \colon P(\mathfrak{G}) \to X$ such that $f_*(u_{\mathfrak{G}}) = [C]$.

(3) What if we choose another matching μ' for the same chain C ? What if we have two homologous chains C, C' with matchings μ, μ' ?