Aufgaben zur Topologie II

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Week 2 — Simplicial schemes due by: 03.05.2017

Exercise 2.1 (Sums and topological products)
Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two simplicial schemes.
a) Define their disjoint sum $\mathfrak{X} \sqcup \mathfrak{Y}$ with inclusions

$$\iota_1 \colon \mathfrak{X} \to \mathfrak{X} \sqcup \mathfrak{Y} \quad \text{and} \quad \iota_2 \colon \mathfrak{Y} \to \mathfrak{X} \sqcup \mathfrak{Y}.$$

Show the universal property and show that there is a homeomorphism

$$|\mathfrak{X} \sqcup \mathfrak{Y}| \cong |\mathfrak{X}| \sqcup |\mathfrak{Y}|.$$

b) Find a simplicial scheme $\mathfrak{Z}$ with simplicial maps $\phi_1 \colon \mathfrak{Z} \to \mathfrak{X}$ and $\phi_2 \colon \mathfrak{Z} \to \mathfrak{Y}$, such that the geometric realisation $|\mathfrak{Z}|$ is homeomorphic to $|\mathfrak{X}| \times |\mathfrak{Y}|$, in such a way that $|\phi_1|$ and $|\phi_2|$ correspond respectively to the projections of $|\mathfrak{X}| \times |\mathfrak{Y}|$ on $|\mathfrak{X}|$ and on $|\mathfrak{Y}|$.

Hint: The n-simplices of $|\mathfrak{Z}|$ are not products $\sigma \times \tau$ of a p-simplex $\sigma = \{v_0, \dots, v_p\}$ in $\mathfrak{X}$ and a q-simplex $\tau = \{w_0, \dots, w_q\}$ in $\mathfrak{Y}$ with $p+q = n$. Rather they are determined by a monotone path $\pi = (\,(a_0, b_0), (a_1, b_1), \dots, (a_n, b_n)\,)$ in $\{0, \dots, p\} \times \{0, \dots, q\}$ of length $n+1$, for some $p+q \geq n$ and some choice of a p-simplex in $\mathfrak{X}$ and a q-simplex in $\mathfrak{Y}$. Remember the decomposition of a prism $\Delta^n \times I$ into $n+1$ simplices $\Delta_0^{n+1}, \dots, \Delta_n^{n+1}$ used to proof the homotopy axiom.

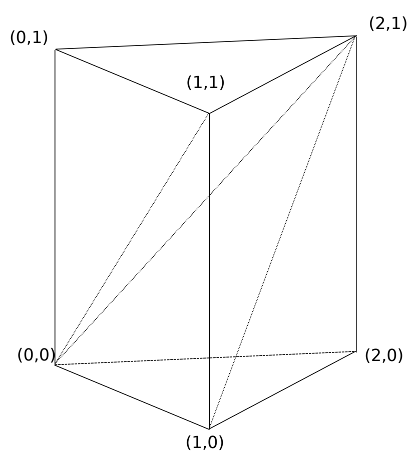

Figure 1: The prism $\Delta^1 \times \Delta^2$ can be dissected into 4 tetrahedra.

Exercise 2.1** (Product of simplicial schemes) Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two simplicial schemes. We address the problem of finding a product $\mathfrak{X} \otimes \mathfrak{Y}$ in the category of simplicial schemes. Hence we want to construct a simplicial scheme $\mathfrak{X} \otimes \mathfrak{Y}$ with simplicial maps, called projections

$$\pi_1 \colon \mathfrak{X} \otimes \mathfrak{Y} \to \mathfrak{X} \quad \text{and} \quad \pi_2 \colon \mathfrak{X} \otimes \mathfrak{Y} \to \mathfrak{Y}$$

satisfying the universal property of products.

a) Let $\mathfrak{X}$ be the standard p-simplex and $\mathfrak{Y}$ the standard q-simplex. Show that $\mathfrak{X} \otimes \mathfrak{Y}$ is isomorphic to the standard $(pq+p+q)$-simplex, and describe the projections.
b) For generic $\mathfrak{X}$ and $\mathfrak{Y}$ construct the simplicial scheme $\mathfrak{X} \otimes \mathfrak{Y}$ with projections π_1 and π_2, and show that the set of vertices of $\mathfrak{X} \otimes \mathfrak{Y}$ is naturally identified with the product of the sets of vertices of $\mathfrak{X}$ and $\mathfrak{Y}$, by looking at the behaviour of the maps π_1 and π_2 on vertices.
c) Recall the simplicial scheme $\mathfrak{Z}$ with maps ϕ_1 and ϕ_2 from the previous exercise. By universal property we can use ϕ_1 and ϕ_2 to define a map $\phi\colon \mathfrak{Z} \to \mathfrak{X} \otimes \mathfrak{Y}$. Show that $|\phi|$ is a homotopy equivalence.

Exercise 2.2 (Simplicial approximation of selfmaps of $\mathbb{S}^1$ and $\mathbb{S}^2$)
a) The space $\mathbb{S}^1$ can be written as a polygon with $n \geq 3$ vertices; this corresponds to a simplicial scheme $\mathfrak{P}(n)$ with n vertices and n 1-simplices. Suppose we want to approximate the degree k map $z \mapsto z^k$ on $\mathbb{S}^1$ by a simplicial map $\varphi\colon \mathfrak{P}(n) \to \mathfrak{P}(n')$. Find n, n' and a simplicial map with degree $\deg(|\varphi|) = k$.

b) Do the same for a map $\mathbb{S}^2 \to \mathbb{S}^2$ of degree k.

c) Consider the five Platonic solids as simplicial schemes for $\mathbb{S}^2$. Which are fine enough to permit the antipodal map as a simplicial map? In case it is not, can you find a homotopy of the antipodal map to a simplicial map ?

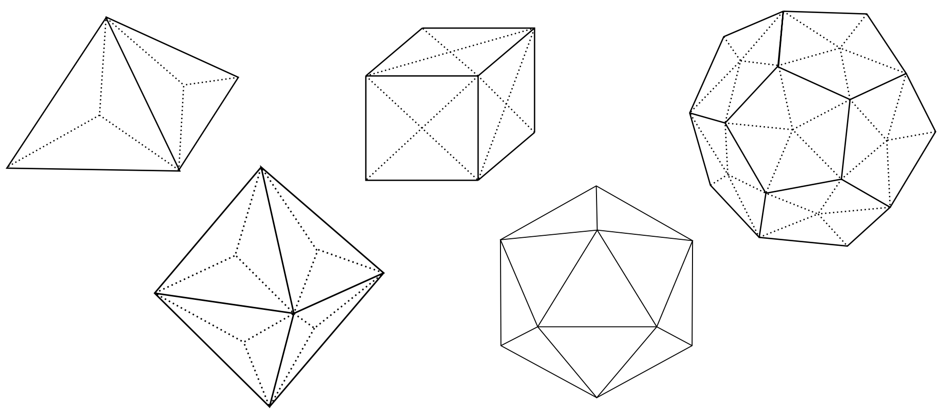

Figure 2: The five Platonic solids realised as simplicial schemes.

Exercise 2.3 (Simplicial chain complex)
Recall how to associate to a CW complex X the cellular chain complex and do the same for a polyhedron $X = |\mathfrak{X}|$.

a) The n-th chain group is $\mathcal{C}_n(\mathfrak{X}) = \mathbb{Z}\langle \sigma \mid \dim(\sigma) = n \rangle$, the free abelian group generated by all n-simplices.

b) The differential $d\colon \mathcal{C}_n(\mathfrak{X}) \to \mathcal{C}_{n-1}(\mathfrak{X})$ is given by

$$d(\sigma) = \sum_{\tau \in \mathfrak{X}_{n-1}} \operatorname{inc}(\sigma, \tau) \cdot \tau$$

where the **incidence number** $\operatorname{inc}(\sigma, \tau)$ of τ in σ is $(-1)^i$ if $\tau = d_i(\sigma)$ for some i; and it is 0 if τ is not a face of σ.

c) Analogously to the natural isomorphism between cellular homology and singular homology, define a natural transformation

$$\vartheta\colon H_*(\mathcal{C}_\bullet(\mathfrak{X})) \cong H_*(|\mathfrak{X}|)$$

which is an isomorphism. (You do not need to proof that ϑ is an isomorphism.)

d) Show that the map ϑ is natural for simplicial maps $\varphi\colon \mathfrak{X} \to \mathfrak{Y}$ (which induce chain maps $\mathcal{C}(\varphi)\colon \mathcal{C}_\bullet(\mathfrak{X}) \to \mathcal{C}_\bullet(\mathfrak{Y})$ on the left-hand side and realizations $|\varphi|\colon |\mathfrak{X}| \to |\mathfrak{Y}|$ on the right-hand side).

Exercise 2.4 (Push-forward and pull-back of a simplicial scheme)
a) Let $f\colon \mathfrak{X}_0 \to \mathfrak{Y}_0$ be a function of sets and let $\mathfrak{X}$ be a simplicial scheme with vertex set $\mathfrak{X}_0$. Proof that there

is a simplicial scheme $f_*(\mathfrak{X})$ with vertex set $\mathfrak{Y}_0$ and a simplicial map $\psi\colon \mathfrak{X} \to f_*(\mathfrak{X})$ with $\psi|_{\mathfrak{X}_0} = f$ that has the following universal property: Any simplicial map $\varphi\colon \mathfrak{X} \to \mathfrak{Y}$ factors as

$$\varphi = \iota \circ \psi\colon \mathfrak{X} \to f_*(\mathfrak{X}) \to \mathfrak{Y}$$

where $f = \varphi|_{\mathfrak{X}_0}\colon \mathfrak{X}_0 \to \mathfrak{Y}_0$ and ι is the canonical inclusion.

b) Dual to this, define a pull-back of a simplicial scheme and find its universal property.

Exercise 2.5*(Elementary collapse)
Let $\mathfrak{X}$ be a simplicial scheme. Given a p-simplex σ of $\mathfrak{X}$, a *free face* of σ is a $(p-1)$-simplex τ such that τ is a face of σ (i.e. $\tau = d_i(\sigma)$ for exactly one i between 0 and p) and the only simplices of $\mathfrak{X}$ containing τ are σ and τ itself.

a) Show that there is a simplicial scheme $\mathfrak{X}'$ whose simplices are exactly those of $\mathfrak{X}$ but σ and τ.

b) Show that the natural inclusion $\mathfrak{X}' \hookrightarrow \mathfrak{X}$ induces a homotopy equivalence $|\mathfrak{X}'| \hookrightarrow |\mathfrak{X}|$ between the two geometric realisations, by proving that $|\mathfrak{X}'|$ is a deformation retract of $|\mathfrak{X}|$.