Aufgaben zur Topologie II

Prof. Dr. C.-F. Bödigheimer

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Week 1 — CW complexes and homology computations due by: 26.04.2017

Exercise 1.1 (Simplicial Approximation I)
Let X be a finite simplicial complex and let Y be a finite or countably infinite simplicial complex. Using the Simplicial Approximation Theorem for maps, show that the set $[X, Y]$ of homotopy classes $f\colon X \to Y$ is at most countably infinite.

Exercise 1.2 (Simplicial Approximation II)
Conclude from exercise 1.1 and the Simplicial Approximation Theorem for CW Complexes: There are exactly countably many homotopy types among the finite CW complexes.

Exercise 1.3 (Acylic Spaces)
A space X is called **acyclic**, if $\tilde{H}_i(X) = 0$ for all $i \geq 0$. In other words, X is connected and $H_i(X) = 0$ for all $i > 0$. We consider the following example of a non-contractible, acyclic space: $X = (\mathbb{S}^1 \vee \mathbb{S}^1) \cup_f (e_1^2 \sqcup e_2^2)$, by attaching to $X_1 = \mathbb{S}^1 \vee \mathbb{S}^1$ two 2-cells e_1^2 resp. e_2^2 with attaching maps f_1 given by the word $a^5 b^{-3}$ resp. f_2 given by the word $b^3(ab)^{-2}$ written in generators $a, b \in \pi_1(X_1) \cong \mathrm{Fr}(a, b)$.

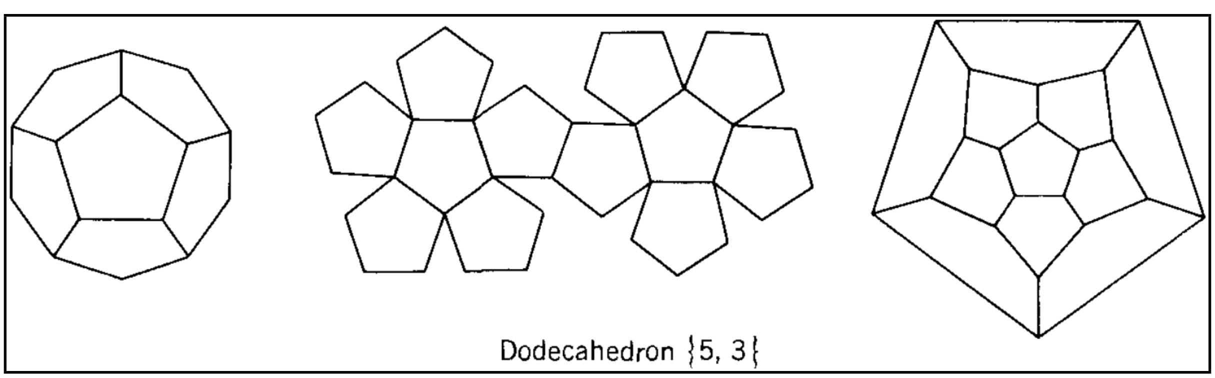
1. Show that X is acyclic by considering the cellular boundary map

$$d\colon C_2(X) \to C_1(X), \quad d = \begin{pmatrix} 5 & -2 \\ -3 & -1 \end{pmatrix}$$

   which is an isomorphism.

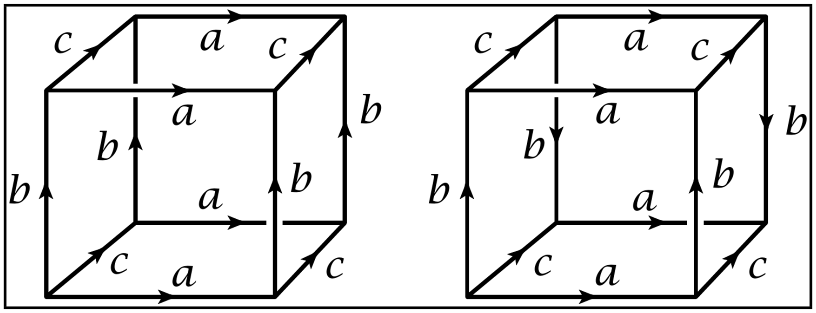
2. Show that X is not contractible.

To show 2., you are allowed to use that $\pi_1(X) = \langle a, b \mid a^5 = b^3 = (ab)^2 \rangle$ surjects onto the symmetry group of the dodecahedron $G < SO(3)$ as follows: Associate to a the rotation ρ_a through an angle of $2\pi/5$ about the axis through a center of the pentagonal face; and associate to b the rotation ρ_b through an angle $2\pi/3$ about the axis through a vertex of a face. The subgroup $G < SO(3)$ generated by ρ_a and ρ_b has 60 elements; the kernel of the map $\rho\colon \pi_1(X) \to G$, $a \mapsto \rho_a$, $b \mapsto \rho_b$ has two elements and is the center of π_1.



Aus Coxeter: Introduction to geometry, Seite 151.

Exercise 1.4 (T^3 and $K \times \mathbb{S}^1$)
Consider the CW complexes X_1 and X_2 shown in the figure below. Both have a single 0-cell, three 1-cells, three 2-cells and a single 3-cell. Note that the CW complex on the left is a model for the three torus T^3 whereas the CW complex on the right is a model for the product of the Klein bottle K and the circle $\mathbb{S}^1$. Compute the homology of X_1 and X_2 using cellular homology only.



Aus Hatcher: Algebraic Topology, Seite 142.

Exercise 1.5* (Simplicial Mapping Cylinder)
Let K and L be simplicial complexes and let $f\colon K \to L$ be a simplicial map. A **simplicial mapping cylinder** of the simplicial map $f\colon K \to L$ is a simplicial complex $M(f)$ that contain both L and the barycentric subdivision K' of K as subcomplexes, and such that there is a deformation retraction r_t of $M(f)$ onto L with $r_1|_{K'} = f$.
Construct a simplicial mapping cylinder for a given map $f\colon K \to L$. For advice consider Hatcher: Algebraic Topology pages 182 to 184.