Aufgaben zur Topologie

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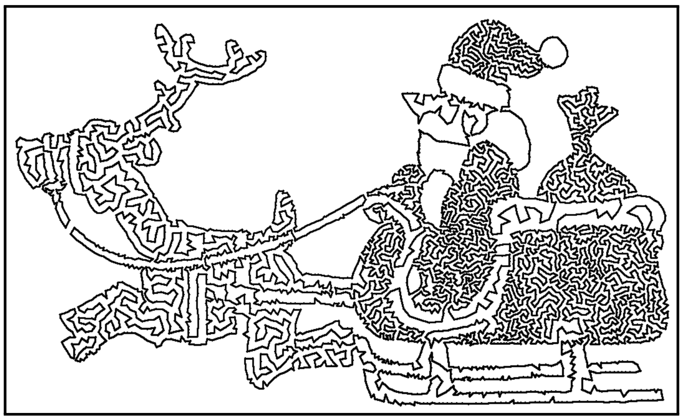
Week 9 — Long exact homology sequences Due: 11. January 2017



Henry Poincaré and his red-nosed reindeer called Rudolph Homology.

Exercise 9.1 (The connecting homomorphism I)
Let

$$0 \to A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \to 0$$

be a short exact sequence of chain complexes over $\mathbb{K}$ and denote by $\partial_*\colon H_n(C_\bullet) \to H_{n-1}(A_\bullet)$ the connecting homomorphism.
(a) ∂_* is a homomorphism of $\mathbb{K}$-modules.
(b) ∂_* is natural.

Exercise 9.2 (Induced maps on homology in degree zero.)
Let X be a space and denote by $\pi_0(X)$ the set of path-components of X.

(1) Show that there is an isomorphism $H_0(X) \cong \mathbb{Z}\langle\pi_0(X)\rangle := \bigoplus_{\pi_0(X)} \mathbb{Z}$, the free abelian group generated by the set $\pi_0(X)$.

(2) Show that any continuous map $f\colon X \to Y$ induces a well-defined function $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$.

(3) If we identify $H_0(X)$ with $\mathbb{Z}\langle\pi_0(X)\rangle$ and $H_0(Y)$ with $\mathbb{Z}\langle\pi_0(Y)\rangle$ as above, then the homomorphisms $H_0(f)$ and $\mathbb{Z}\langle\pi_0(f)\rangle$ are equal.

Exercise 9.3 (Relative homology in degree zero.)
Let A be a subspace of X; we denote by $i\colon A \to X$ the inclusion and by $q\colon X \to X/A$ the quotient map identifying A to a point. (What does this mean, when $A = \emptyset$?). For the sets of path-components we have induced functions $\pi_0(i)\colon \pi_0(A) \to \pi_0(X)$ and $\pi_0(q)\colon \pi_0(X) \to \pi_0(X/A)$.

(1) Show that the relative homology group $H_0(X, A)$ is isomorphic to $\mathbb{Z}\langle\pi_0(X/A)\rangle/\mathbb{Z}\langle[A/A]\rangle$, where $[A/A]$ denotes the path-component of X/A containing the point A/A.

(2) When A is non-empty, show that $H_0(q)\colon H_0(X) \to H_0(X, A)$ is surjective and that $H_0(X, A)$ is isomorphic to

the quotient module $H_0(X)/\mathrm{im}(H_0(i))$.
(3) Describe $H_1(X, A)$ and the connecting homomorphism $\partial_* \colon H_1(X, A) \to H_0(A)$.

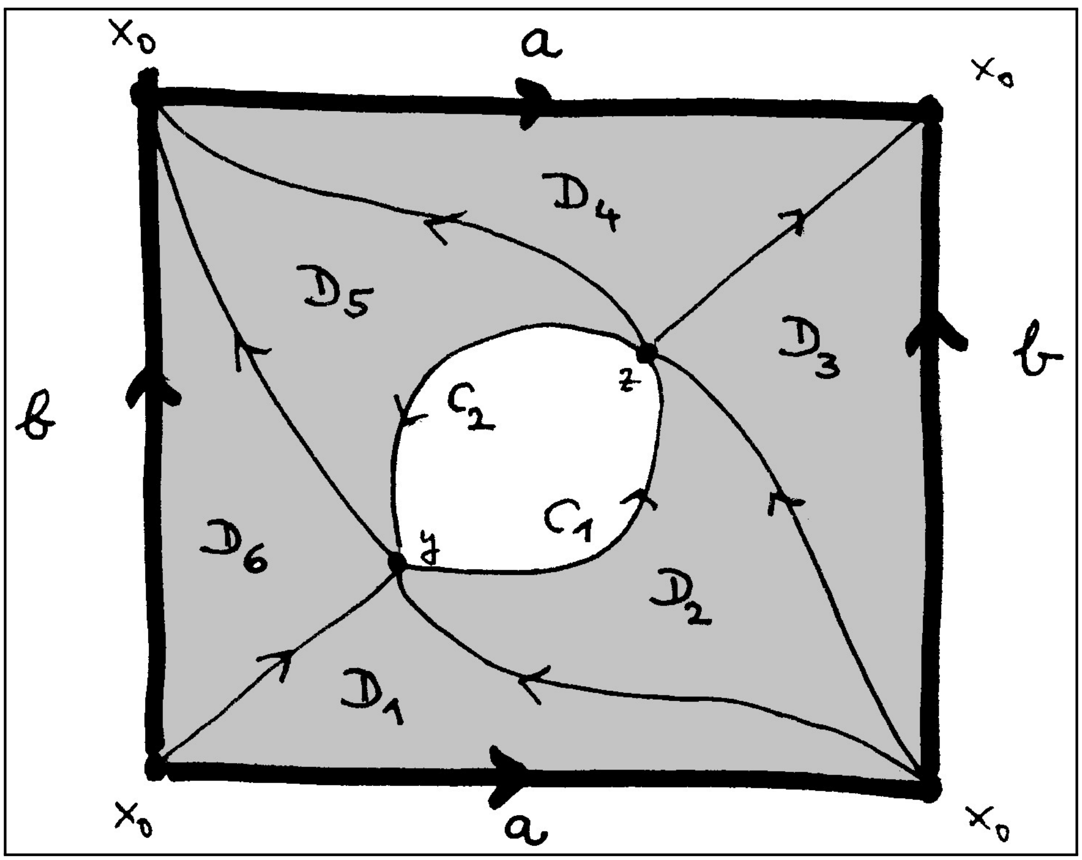
Exercise 9.4 (Torus with one boundary curve.)
Let X be a torus with one boundary curve, as shown in the figure. We denote by A the boundary curve, by $i \colon A \to X$ the inclusion of spaces, by $S_\bullet(i) \colon S_\bullet(A) \to S_\bullet(X)$ the inclusion of singular chain complexes, and by $q_\bullet \colon S_\bullet(X) \to S_\bullet(X, A) = S_\bullet(X)/S_\bullet(A)$ the quotient homomorphism of singular chain complexes; thus we have the short exact sequence of chain complexes

$$0 \to S_\bullet(A) \xrightarrow{S_\bullet(i)} S_\bullet(X) \xrightarrow{q_\bullet} S_\bullet(X, A) \to 0$$

leading to a long exact sequence of homology groups:

$$\ldots \to H_2(A) \xrightarrow{i_*} H_2(X) \xrightarrow{q_*} H_2(X, A) \xrightarrow{\partial_*} H_1(A) \xrightarrow{i_*} H_1(X) \xrightarrow{q_*} H_1(X, A) \xrightarrow{\partial_*} H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{q_*} H_0(X, A) \to 0$$

(1) Compute all homology groups in degrees 0 and 1 and all homomorphisms between them.
(2) Show that $H_2(X, A)$ contains at least a summand which is isomorphic to $\mathbb{Z}$ and generated by the relative cycle $D = -D_1 + D_2 + D_3 + D_4 + D_5 - D_6$.
(3) Compute for the connecting homomorphism $\partial_*([D]) = [c_1 + c_2]$.



A torus, written as a square with dark edges a and b identified. Some singular 1- and 2-simplices, used in Exercise 9.4, are shown.

Exercise 9.5 (Calculating with long exact sequences of abelian groups.)

(1) Show that, if $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \to 0$ is an exact sequence, then we also have exact sequences:

(a) $0 \to B/\alpha(A) \xrightarrow{\bar{\beta}} C \xrightarrow{\gamma} D \to 0$ and

(b) $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} \text{image}(\beta) \to 0$.

(2) More generally, if we have an exact sequence $\cdots \to A \xrightarrow{e} B \xrightarrow{f} C \xrightarrow{g} D \xrightarrow{h} E \to \cdots$, then we may "localise" it at C, meaning that there is a short exact sequence:

$$0 \to \text{coker}(e) \xrightarrow{\bar{f}} C \xrightarrow{g} \ker(h) \to 0,$$

where $\text{coker}(e)$ means $B/e(A)$.
(3) Show that, if $0 \to A \to B \to \mathbb{Z}^k \to 0$ is exact, then B is isomorphic to $\mathbb{Z}^k \oplus A$.
(Hint: start by constructing a homomorphism $\mathbb{Z}^k \to B$, which is a right-inverse for the given homomorphism $B \to \mathbb{Z}^k$.)
(4) Give an example of a short exact sequence $0 \to A \to B \to \mathbb{Z}/k\mathbb{Z}$ (for any $k \neq -1, 0, +1$) for which B is not isomorphic to $A \oplus \mathbb{Z}/k\mathbb{Z}$.

Exercise 9.6* (Connecting homomorphisms II)
Assume we have a short exact sequence

$$(*) \quad 0 \to A_\bullet \xrightarrow{\alpha_\bullet} B_\bullet \xrightarrow{\beta_\bullet} C_\bullet \to 0$$

of free chain complexes; we know that $B_n \cong A_n \oplus C_n$, since C_n is free. But the boundary operator $\partial^B\colon B_n \to B_{n-1}$ may not be the direct sum of ∂^A und ∂^C, i.e., not of diagonal block form; but it must have the form

$$\partial^B = \begin{pmatrix} \partial^A & \varphi_n \\ 0 & \partial^C \end{pmatrix}$$

for some family of homomorphisms $\varphi_n\colon C_n \to A_{n-1}$.
(1) Which condition must this family satisfy, such that $\partial^B \circ \partial^B = 0$?
(2) Compute the connecting homomorphism in the long exact homology sequence of (*).