Aufgaben zur Topologie

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Week 8 — Homological algebra and homotopy invariance of homology Due: 21. December 2016

> § 5. — Homologies.
>
> Considérons une variété V à p dimensions; soit maintenant W une variété à q dimensions ($q \leq p$) faisant partie de V. Supposons que la frontière complète de W se compose de λ variétés continues à $q-1$ dimensions
>
> $$v_1, \quad v_2, \quad \dots, \quad v_\lambda.$$
>
> Nous exprimerons ce fait par la notation
>
> $$v_1 + v_2 + \dots + v_\lambda \backsim 0.$$
>
> Plus généralement la notation
>
> $$k_1 v_1 + k_2 v_2 \backsim k_3 v_3 + k_4 v_4,$$
>
> où les k sont des entiers et les v des variétés à $q-1$ dimensions, signifiera qu'il existe une variété W à q dimensions faisant partie de V et dont la frontière complète se composera de k_1 variétés peu différentes de v_1, de k_2 variétés peu différentes de v_2, de k_3 variétés peu différentes de la variété opposée à v_3 et de k_4 variétés peu différentes de la variété opposée à v_4.
>
> Les relations de cette forme pourront s'appeler des *homologies*.

The birth of homology, from *Analysis Situs*, H. Poincaré (1895).

Exercise 8.1 (The five-lemma.)
Prove the famous *five-lemma*. Let R be a ring and suppose we have the following diagram of modules over R:

$$\begin{array}{ccccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\varepsilon} \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E'
\end{array}$$

Assume that this diagram is commutative and that the two horizontal rows of homomorphisms are exact. Moreover, assume that α is surjective, β and δ are bijective and ε is injective. Prove that γ is bijective.

Application: let $\phi\colon B \to B'$ be a homomorphism of R-modules taking a submodule $A \subseteq B$ to a submodule $A' \subseteq B'$, so that we have restricted and induced homomorphisms $\phi|_A\colon A \to A'$ and $\bar{\phi}\colon B/A \to B'/A'$. If $\phi|_A$ and $\bar{\phi}$ are both isomorphisms then so is ϕ.

Exercise 8.2 (Mapping cones and mapping cylinders of chain complexes.)
Let A and B be chain complexes with differential ∂_A resp. ∂_B and let $f\colon A \to B$ be a chain map.
(i) Define a new chain complex $\mathrm{Cone}(f)$ by $\mathrm{Cone}(f)_n = A_{n-1} \oplus B_n$ and setting its differential $\partial\colon A_{n-1} \oplus B_n \to A_{n-2} \oplus B_{n-1}$ to be the sum of the four maps

$$\partial_A\colon A_{n-1} \to A_{n-2} \qquad \partial_B\colon B_n \to B_{n-1} \qquad 0\colon B_n \to A_{n-2} \qquad (-1)^n.f_{n-1}\colon A_{n-1} \to B_{n-1},$$

or as a formula

$$\partial(a,b) := (\partial_A(a), (-1)^n.f_{n-1}(a) + \partial_B(b)).$$

(1) Prove that this is indeed a chain complex.
(2) Construct chain maps $B \to \mathrm{Cone}(f)$ and $\mathrm{Cone}(f) \to A[1]$, where $A[1]$ simply means the chain complex A with the modified grading $A[1]_n = A_{n-1}$, and show that you have constructed a short exact sequence

$$0 \to B \longrightarrow \mathrm{Cone}(f) \longrightarrow A[1] \to 0.$$

(ii) Now define a chain complex $\mathrm{Cyl}(f)$ by $\mathrm{Cyl}(f)_n = A_{n-1} \oplus B_n \oplus A_n$ with differential $\partial\colon \mathrm{Cyl}(f)_n \to \mathrm{Cyl}(f)_{n-1}$ given in block form by the matrix

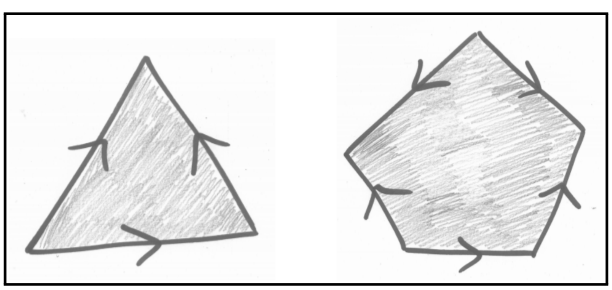
$$\begin{pmatrix} \partial_A & 0 & 0 \\ (-1)^n.f_{n-1} & \partial_B & 0 \\ (-1)^n.\mathrm{id} & 0 & \partial_A \end{pmatrix}.$$

(3) Prove that this is a chain complex.
(4) Construct a chain homotopy equivalence $\mathrm{Cyl}(f) \simeq B$.

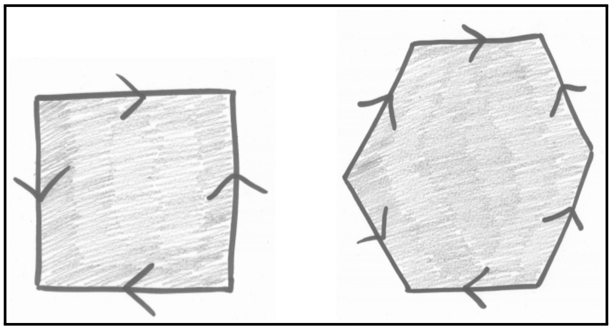
Exercise 8.3 (Cones of continuous maps and dunce caps.)
Let Z be a space with subspace $A \subseteq Z$ and let $f\colon A \to Y$ be a continuous map. Recall (cf. Exercise 5.6) that the space $Z \cup_f Y$ is defined to be the quotient of the disjoint union $Z \sqcup Y$ by the smallest equivalence relation $\sim$ such that $a \sim f(a)$ for all $a \in A$.
Now let $g\colon X \to Y$ be a continuous map and define its *mapping cylinder* to be $\mathrm{Cyl}(g) = Z \cup_f Y$ where $Z = X \times [0,1]$, $A = X \times \{0\}$ and f is g composed with the obvious identification $X \times \{0\} \cong X$. Define its *mapping cone* to be $\mathrm{Cone}(g) = \mathrm{Cyl}(g)/{\sim}$, where $\sim$ is the smallest equivalence relation such that $(x,1) \sim (x',1)$ for all $x, x' \in X$.

(1) Draw a picture to show what is going on geometrically in these constructions.
(2) Construct an embedding $Y \to \mathrm{Cone}(g)$ and a projection $\mathrm{Cone}(g) \to \Sigma X$, where ΣX is the *suspension* of X, defined to be $X \times [0,1]/{\sim}$, where $\sim$ is the smallest equivalence relation such that $(x,1) \sim (x',1)$ and $(x,0) \sim (x',0)$ for all $x, x' \in X$.
(3) Show that $\mathrm{Cyl}(g)$ is homotopy equivalent to Y.
(We will see later in the course that these constructions yield those of the previous exercise after applying the singular chain functor.)

(4) Now let $A \subseteq X$ be a closed subspace for which there exists an open neighbourhood $U \supseteq A$ that deformation retracts onto A. Let $f, g\colon A \to Y$ be two continuous maps which are homotopic. Prove that $X \cup_f Y$ and $X \cup_g Y$ are homotopy equivalent. Thus the operations $\mathrm{Cyl}(-)$ and $\mathrm{Cone}(-)$ are homotopy invariant.
(5) Thus show that the following two spaces are contractible:



(Hint: realise each of them as $\mathbb{D}^2 \cup_f \mathbb{S}^1$ for some map $f\colon \partial\mathbb{D}^2 \to \mathbb{S}^1$, and consider the degree of this map.)
The left-hand space above is often called the "dunce cap". There are many *generalised dunce caps* like the right-hand space above – each of them is the quotient of a polygon with an odd number of sides, which are all identified with certain choices of orientations.
(6) Using a similar trick to above, show that the following two spaces each have fundamental group isomorphic to $\mathbb{Z}$, and draw a generator in each case:



Exercise 8.4 (Mapping tori of chain complexes.)
Let R be a ring, C a chain complex of R-modules and $f\colon C \to C$ be a chain map from C to itself. We can formally adjoin an invertible indeterminate t to C to obtain a chain complex $\bar{C}$ of $R[t^{\pm 1}]$-modules by first setting $\bar{C}_n = C_n \otimes_R R[t^{\pm 1}]$ and then defining $\bar{\partial}$ to be ∂ extended by linearity in t (more formally: $\bar{\partial} = \partial \otimes \mathrm{id}$, where id is the identity map $R[t^{\pm 1}] \to R[t^{\pm 1}]$). Here, $R[t^{\pm 1}]$ is the ring of *Laurent polynomials* in t with coefficients in R, or, equivalently, the *group-ring* $R[\mathbb{Z}]$ of the group $\mathbb{Z}$ with coefficients in R. The chain map f extends by linearity in t to a chain map $\bar{f}\colon \bar{C} \to \bar{C}$. There is also a canonical chain map $t\colon \bar{C} \to \bar{C}$ where each $t_n\colon \bar{C}_n \to \bar{C}_n$ is just multiplication by t. Define:

$$\mathrm{Torus}(f) = \mathrm{Cone}(\bar{f} - t).$$

(1) Describe this explicitly in terms of the C_n, ∂_C and f_n.
(2) Suppose that we have a commutative square

$$\begin{array}{ccc} C & \xrightarrow{f} & C \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ D & \xrightarrow{g} & D \end{array}$$

Define a chain map $(\alpha, \beta)_\sharp\colon \mathrm{Torus}(f) \to \mathrm{Torus}(g)$ and show that your construction satisfies the two functoriality properties $(\mathrm{id}, \mathrm{id})_\sharp = \mathrm{id}$ and $(\alpha, \beta)_\sharp \circ (\alpha', \beta')_\sharp = (\alpha \circ \alpha', \beta \circ \beta')_\sharp$.
(3) Show that, for any chain map $f\colon C \to C$, the chain map $(f, f)_\sharp\colon \mathrm{Torus}(f) \to \mathrm{Torus}(f)$ induces isomorphisms on all homology groups.
(Hint: construct a chain homotopy from $(f, f)_\sharp$ to the "multiplication by t" chain map from $\mathrm{Torus}(f)$ to itself. Then show that this "multiplication by t" chain map induces isomorphisms on all homology groups and use the homotopy-invariance property of homology to deduce that the same is true for $(f, f)_\sharp$.)
(4) Deduce that, for chain maps $f\colon C \to D$ and $g\colon D \to C$, the chain complexes $\mathrm{Torus}(f \circ g)$ and $\mathrm{Torus}(g \circ f)$ have the same homology groups.
(Hint: consider the chain maps $(f \circ g, f \circ g)_\sharp$ and $(g \circ f, g \circ f)_\sharp$.)

> **The 6-point space Σ^2 "would be homeomorphic to a 2-sphere if it were only Hausdorff." More precisely, consider the following conditions on a topological space X : (1) *The complement of each point in X is acyclic* (*in singular homology*); (2) $H_2(X) \neq 0$. We have seen that the T_0 space Σ^2 satisfies these two conditions. However, simply by adding the extra condition (3) *X is Hausdorff*, one can conclude that X is homeomorphic to the 2-sphere. (See [5].)**

From *Singular homology groups and homotopy groups of finite topological spaces* by M. C. McCord (1966). In his notation, Σ^2 is the 6-point space considered in Exercise 8.5(c) on the next page. Thus you have a strong hint as to what the homology of that space "should" be!

Exercise 8.5 (Finite topological spaces.)
(a) There are three topological spaces X having exactly two points. In each case, compute the singular chain complex $S_\bullet(X)$ and the homology $H_n(X)$ for all n.
(b) Consider the 4-point topological space $\{a, b, c, d\}$ whose topology is generated by the base

$$\{a\}; \{b\}; \{a, b, c\}; \{a, b, d\}$$

and calculate its homology.
(c)* Do the same for the 6-point space $\{a, b, c, d, e, f\}$ whose topology is generated by the base

$$\{a\}; \{b\}; \{a, b, c\}; \{a, b, d\}; \{a, b, c, d, e\}; \{a, b, c, d, f\}.$$

(d)* In general, there is a $(2n + 2)$-point space $\{a_1, b_1, \ldots, a_{n+1}, b_{n+1}\}$ whose topology is generated by the base

$$\{a_1\}; \{b_1\}; \{a_1, b_1, a_2\}; \{a_1, b_1, b_2\};\ \ldots\ldots\ ; \{a_1, b_1, \ldots, a_n, b_n, a_{n+1}\}; \{a_1, b_1, \ldots, a_n, b_n, b_{n+1}\}.$$

Make a conjecture about its homology, and about which (more familiar!) space it is homotopy equivalent to.
(e)** Prove your conjecture.

Exercise 8.6* (A chain complex of chain maps.)
Let C and D be chain complexes of R-modules. We define a *chain map of degree* d to be a collection $f = \{f_n\}_{n\in\mathbb{Z}}$ of homomorphisms of R-modules $f_n\colon C_n \to D_{n+d}$ such that $\partial^D_{n+d} \circ f_n = f_{n-1} \circ \partial^C_n$ for all n. We define a *pre-chain map of degree* d to be simply a collection $f = \{f_n\}_{n\in\mathbb{Z}}$ of homomorphisms $f_n\colon C_n \to D_{n+d}$, with no condition.
(1) Show that the set of all pre-chain maps of a fixed degree d forms an R-module, denoted $\mathrm{PreChain}_d(C, D)$.
(2) Given $f = \{f_n\} \in \mathrm{PreChain}_d(C, D)$, show that the formula

$$(df)_n = \partial^D_{n+d} \circ f_n - (-1)^d f_{n-1} \circ \partial^C_n$$

defines a pre-chain map $df \in \mathrm{PreChain}_{d-1}(C, D)$.
(3) Show that $ddf = 0$, and hence that $\mathrm{PreChain}_\bullet(C, D)$ is a chain complex.
(4) Prove that there is a natural isomorphism between $H_0(\mathrm{PreChain}_\bullet(C, D))$ and the set of chain-homotopy-classes of chain maps (of degree 0) from C to D. (Hint: first note that $Z_0(\mathrm{PreChain}_\bullet(C, D))$ is naturally isomorphic to the set of chain maps of degree 0.)