Aufgaben zur Topologie

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Week 7 — Chain complexes Due: 14. December 2016

Exercise 7.1 (Decomposition of chain complexes.)
(1) Let

$$0 \leftarrow C_0 \leftarrow C_1 \leftarrow \cdots \leftarrow C_{N-1} \leftarrow C_N \leftarrow 0$$

be a bounded chain complex of finitely generated free abelian groups. Show that it splits as a direct sum of finitely many subcomplexes, each of which is of the form

$$0 \leftarrow \mathbb{Z} \leftarrow 0 \qquad \text{or} \qquad 0 \leftarrow \mathbb{Z} \xleftarrow{k} \mathbb{Z} \leftarrow 0$$

for some non-zero $k \in \mathbb{Z}$, up to shifts to the left and right.
(Hint: Use the Elementarteilersatz (Smith normal form) for integer matrices.)

(2) Show that, if we had started with a bounded chain complex of finite-dimensional vector spaces over a field $\mathbb{K}$ instead, then it splits as a direct sum of finitely many subcomplexes of just two types, namely $0 \leftarrow \mathbb{K} \leftarrow 0$ and $0 \leftarrow \mathbb{K} \xleftarrow{\mathrm{id}} \mathbb{K} \leftarrow 0$.

(3) Thus any bounded chain complex of finite-dimensional vector spaces is isomorphic to one with chain modules of the form $C_n = B_n \oplus H_n \oplus B_{n-1}$, where B_n denotes the boundaries of degree n, and where the boundary operator

$$\partial\colon C_n = B_n \oplus H_n \oplus B_{n-1} \twoheadrightarrow B_{n-1} \hookrightarrow B_{n-1} \oplus H_{n-1} \oplus B_{n-2} = C_{n-1}$$

is the projection of C_n onto B_{n-1} composed with the inclusion of B_{n-1} into C_{n-1}. It follows that the homology is $H_n(C_\bullet) \cong H_n$.

Exercise 7.2 (Homology of some small chain complexes.)
(a) Compute the homology of each of the following chain complexes.
(b) Take the tensor product with $\mathbb{Q}$ and compute the homology of the resulting chain complex.
(c) Take the tensor product with $\mathbb{F}_p$ (for a prime p) and compute the homology of the resulting chain complex.

$$(A_\bullet) \qquad 0 \to \mathbb{Z}^2 \xrightarrow{\left(\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix}\right)} \mathbb{Z}^2 \xrightarrow{\left(\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix}\right)} \cdots \xrightarrow{\left(\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix}\right)} \mathbb{Z}^2 \xrightarrow{\left(\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix}\right)} \mathbb{Z}^2 \to 0.$$

$$(B_\bullet) \qquad 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0.$$

$$(C_\bullet) \qquad 0 \to \mathbb{Z}^4 \xrightarrow{\left(\begin{smallmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{smallmatrix}\right)} \mathbb{Z}^6 \xrightarrow{\left(\begin{smallmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & -1 & -1 \end{smallmatrix}\right)} \mathbb{Z}^2 \to 0.$$

(d) Use Exercise 7.1 part (1) to show that (in general – not just for these examples) the computations in (b) and (c) may in fact be deduced directly from the computations in (a), without knowledge of the original chain complex.

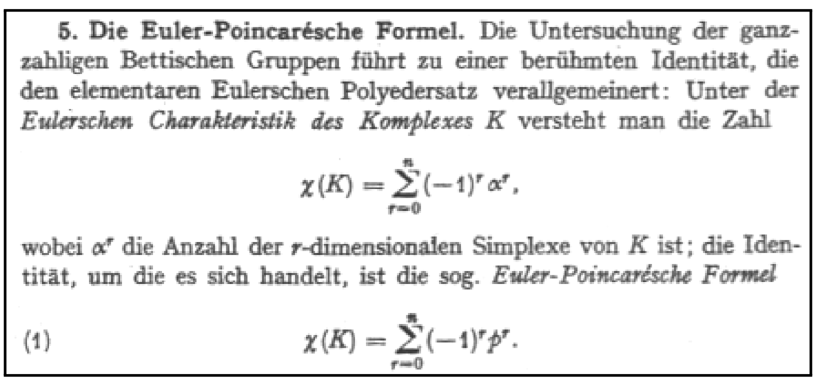
Exercise 7.3 (Chain homotopy is an equivalence relation.)
Recall that a *chain homotopy* $h\colon f_0 \simeq f_1$ between two chain maps $f_0, f_1\colon C_\bullet \to D_\bullet$ is a collection of homomorphisms

$h_n\colon C_n \to D_{n+1}$ such that $h_{n-1} \circ \partial + \partial' \circ h_n = f_0 - f_1$.
Suppose that $C_\bullet, D_\bullet, E_\bullet$ are chain complexes, $f_0, f_1, f_2\colon C_\bullet \to D_\bullet$ and $g_0, g_1\colon D_\bullet \to E_\bullet$ are chain maps and $h\colon f_0 \simeq f_1$, $\hat{h}\colon f_1 \to f_2$ and $k\colon g_0 \simeq g_1$ are chain homotopies. Show that there are chain homotopies
(a) $f_0 \simeq f_0$,
(b) $f_1 \simeq f_0$,
(c) $f_0 \simeq f_2$,
(d) $g_0 f_0 \simeq g_1 f_1$,
given by 0, $-h$, $h + \hat{h}$ and $g_0 h + k f_1$ respectively. Conclude that chain homotopy is an equivalence relation and is preserved by composition.

Exercise 7.4 (Tensor products of chain complexes)
Let $\mathbb{K}$ denote a principal ideal domain. For two chain complexes $A_\bullet$ and $B_\bullet$ over $\mathbb{K}$ with boundary operator ∂^A resp. ∂^B we define a new complex $C_\bullet = A_\bullet \otimes B_\bullet$ by setting $C_n := \sum_{n=k+l} A_k \otimes B_l$ and defining the boundary operator $\partial^\otimes\colon C_n \to C_{n-1}$ by setting (Leibniz-like)

$$\partial^\otimes(a \otimes b) := \partial^A(a) \otimes b \; + \; (-1)^k a \otimes \partial^B(b)$$

for a generator $a \in A_k$ and $b \in B_l$ with $k + l = n$.
(1) Show that this is a chain complex.
(2) Assume $A_n = B_n = 0$ for $n < 0$ and $A_0 = B_0 = \mathbb{K}$. Can you define chain maps $\iota_A\colon A_\bullet \to C_\bullet$ and $\iota_B\colon B_\bullet \to C_\bullet$ and $\pi_A\colon C_\bullet \to A_\bullet$ and $\pi_B\colon C_\bullet \to B_\bullet$ such that $\pi_A \circ \iota_A = \mathrm{id}$, $\pi_B \circ \iota_B = \mathrm{id}$, and $\pi_A \circ \iota_B = 0 = \pi_B \circ \iota_A$?



5. Die Euler-Poincarésche Formel. Die Untersuchung der ganzzahligen Bettischen Gruppen führt zu einer berühmten Identität, die den elementaren Eulerschen Polyedersatz verallgemeinert: Unter der *Eulerschen Charakteristik des Komplexes* K versteht man die Zahl

$$\chi(K) = \sum_{r=0}^{n} (-1)^r \alpha^r,$$

wobei α^r die Anzahl der r-dimensionalen Simplexe von K ist; die Identität, um die es sich handelt, ist die sog. *Euler-Poincarésche Formel*

(1) $$\chi(K) = \sum_{r=0}^{n} (-1)^r p^r.$$

Figure 1: From *Topologie I*, by P. Alexandroff and H. Hopf (1935), page 214. What they term *Betti groups* are the homology groups with integer coefficients of a space. In the first formula, α^r is the number of r-dimensional simplices of the n-dimensional simplicial complex K (cf. Exercise 7.6 on page 4), whereas, in the second formula, p^r denotes the *rank* of the r-th homology group $H_r(K;\mathbb{Z})$. (The *rank* of an abelian group is defined exactly analogously to the *dimension* of a vector space.)

Exercise 7.5 (Euler characteristic)
If $C_\bullet$ is a bounded chain complex of finite-dimensional vector spaces over a field $\mathbb{K}$, we can define its *Euler characteristic* by

$$\chi(C_\bullet) := \sum_n (-1)^n \dim_{\mathbb{K}}(C_n).$$

Show the following formulae:
(1) $\chi(C_\bullet) = \sum_n (-1)^n \dim_{\mathbb{K}} H_n(C_\bullet)$. (Hint: use Exercise 7.1 part (3).)
(2) $\chi(A_\bullet \oplus B_\bullet) = \chi(A_\bullet) + \chi(B_\bullet)$.
(3) $\chi(A_\bullet \otimes B_\bullet) = \chi(A_\bullet)\,\chi(B_\bullet)$.

(Comment to (1): This is the famous formula of Euler-Poincaré-Hopf (see Figure 1 above): the Euler characteristic depends only on the homology. This is true in general, not only over fields, as we will see later.)

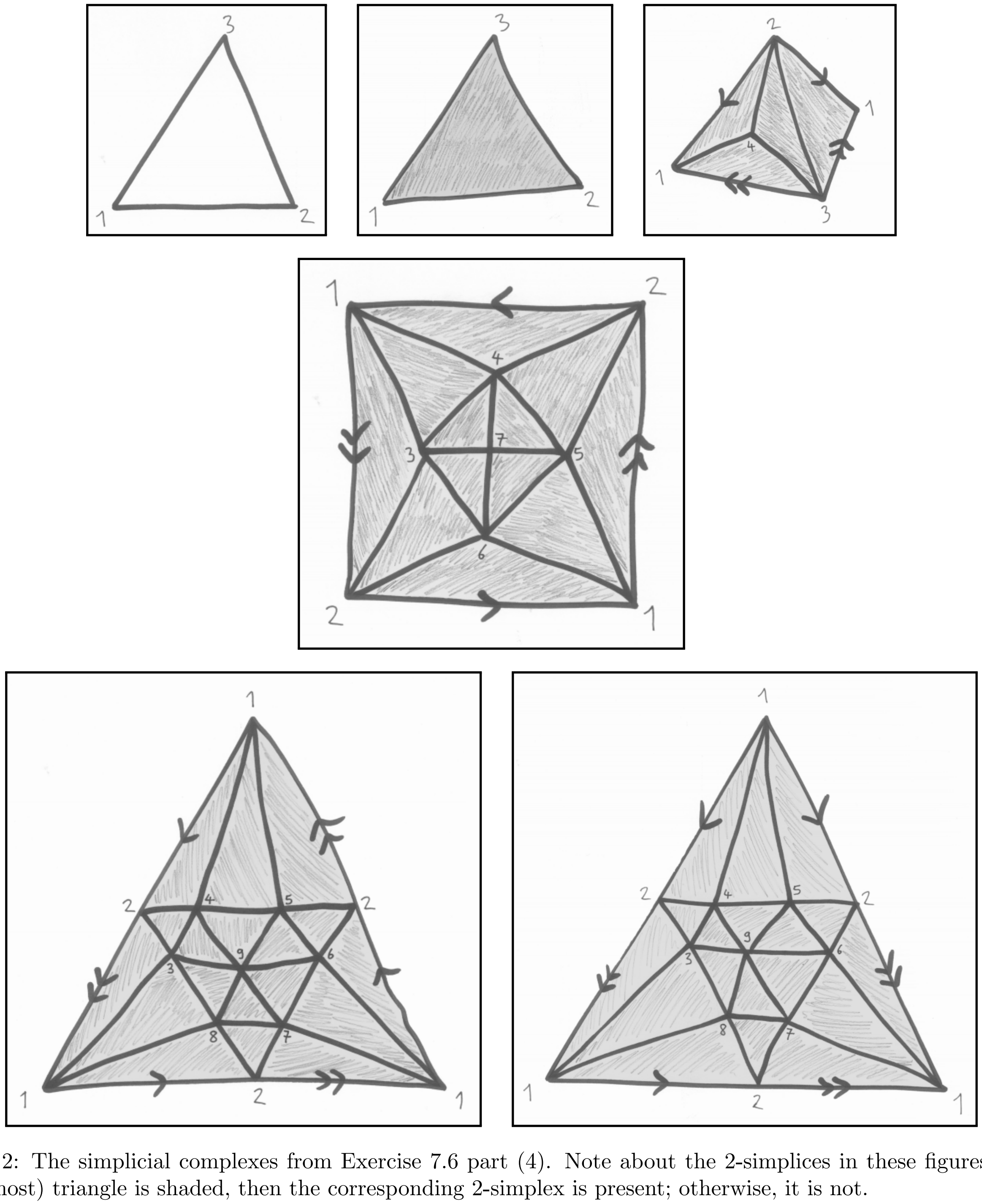


Figure 2: The simplicial complexes from Exercise 7.6 part (4). Note about the 2-simplices in these figures: if an (innermost) triangle is shaded, then the corresponding 2-simplex is present; otherwise, it is not.

Exercise 7.6* (Simplicial chain complexes)
Let $\mathcal{X}$ be a set of non-empty, finite subsets of some fixed set X_0 such that $\sigma \in \mathcal{X}$ implies $\tau \in \mathcal{X}$ for any non-empty subset τ of σ. We denote by X_n all elements σ of $\mathcal{X}$ with exactly $n+1$ elements of X_0. There is an obvious reason why we call the elements of X_0 *vertices*, the elements of X_1 *edges* or *1-simplices*, those in X_2 *triangles* or *2-simplices* and so on. We assume that X_0 is a linearly ordered set; thus any $\sigma \in X_n$ is an ordered set of $n+1$ vertices, which we number $v_0 < v_1 < \ldots < v_n$ from 0 to n. Denote now, for $i = 0, 1, \ldots, n$, by $d_i'(\sigma)$ the set σ with its i-th element v_i removed; this defines functions $d_i' \colon X_n \to X_{n-1}$ for $n > 0$.
(1) Show that $d_i' \circ d_i' = d_i' \circ d_{i+1}'$ and for $i < j$ that $d_i' \circ d_j' = d_{j-1}' \circ d_i'$.
(2) Denote by $C_n(\mathcal{X})$ the free module over the principal ideal domain $\mathbb{K}$ generated by the set X_n. Consider the homomorphisms $d_i \colon C_n(\mathcal{X}) \to C_{n-1}(\mathcal{X})$ determined by d_i' by linear extension. Show that the formulae from (1) hold also for the d_i.
(3) If we set $\partial := \sum_{i=0}^{n} (-1)^i d_i$ show that $\partial \circ \partial = 0$ holds.
(4) In each of the examples depicted in Figure 2 on the previous page, the figure depicts a *triangulation* of a certain space. The vertices of the triangulation form the set X_0 and a subset σ of X_0 belongs to $\mathcal{X}$ if and only if there exists a simplex (in the figures, this means either an edge, a triangle or a vertex) whose vertices are precisely the vertices corresponding to σ. In each case, write down the chain complex $C_\bullet(\mathcal{X})$ and compute its homology groups $H_n(C_\bullet(\mathcal{X}))$ for all n and its Euler characteristic.