# Aufgaben zur Topologie

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**Week 6 — Fundamental groups and first homology groups** Due: 7. December 2016

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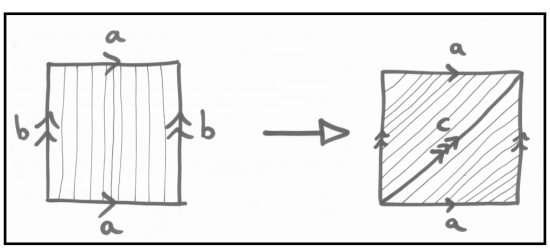
**Exercise 6.1** (Gradients, rotation and divergence: grad, rot and div.)
(1) Let $X$ be an open subset of $\mathbb{R}^2$ and define real vector spaces as follows.
◦ $C_0 = C^\infty(X)$, the space of smooth real-valued functions on $X$.
◦ $C_{-1} = C^\infty(X) \times C^\infty(X)$, to be thought of as the space of smooth vector fields $v = (v_1, v_2)$ on $X$, in coordinates.
◦ $C_{-2} = C^\infty(X)$, to be thought of as the space of volume forms on $X$.
There are linear maps $\mathrm{grad}\colon C_0 \to C_{-1}$ and $\mathrm{rot}\colon C_{-1} \to C_{-2}$ defined by $\mathrm{grad}(f) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ and $\mathrm{rot}(v_1, v_2) = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}$.
(a) Show (using vector calculus) that this is a chain complex (over the field $\mathbb{R}$), where all undefined chain modules are 0.
(b) For arbitrary $X$, find the dimension of $H_0(C_\bullet)$, i.e., the kernel of grad.
(c) When $X$ is not simply-connected, give an example of a vector field that has zero rotation, but is not the gradient of any smooth function on $X$, thus showing that the homology group $H_{-1}(C_\bullet)$ is non-trivial in this case.

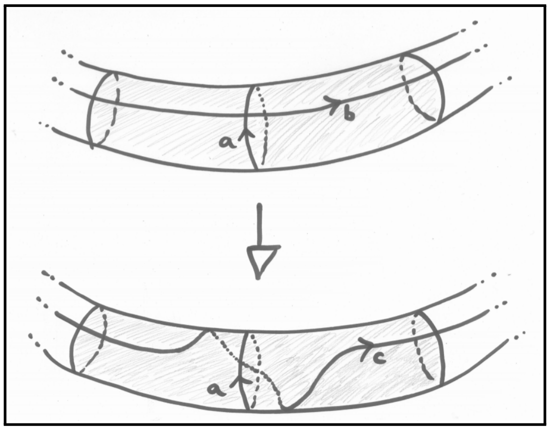
(2) Now let $X$ be an open subset of $\mathbb{R}^3$ and define real vector spaces as follows.
◦ $C_0 = C^\infty(X)$.
◦ $C_{-1} = C^\infty(X) \times C^\infty(X) \times C^\infty(X)$.
◦ $C_{-2} = C^\infty(X) \times C^\infty(X) \times C^\infty(X)$.
◦ $C_{-3} = C^\infty(X)$.
(a) Recall the definitions of the linear operators $\mathrm{grad}\colon C_0 \to C_{-1}$, $\mathrm{rot}\colon C_{-1} \to C_{-2}$ and $\mathrm{div}\colon C_{-2} \to C_{-3}$ in this setting, and show that these form a chain complex.
(b) As above, compute the dimension of $H_0(C_\bullet)$.
(c) Show that the homology group $H_{-1}(C_\bullet)$ is non-trivial when $X$ is not simply-connected by finding a vector field with zero rotation and which is not the gradient of any smooth function on $X$.
(d)* Find an $X$ such that $H_{-2}(C_\bullet)$ is non-trivial, i.e., we need a vector field defined on $X$ with zero divergence and which is not the rotation of any other vector field on $X$. (A first case to consider is $X = \mathbb{R}^3 - \{(0,0,0)\}$.)

**Exercise 6.2** (Induced maps on $\pi_1$ and the abelianisation of $\pi_1$.)
Recall: If $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ is a map of degree $k$, then the induced map $\pi_1(f)\colon \pi_1(\mathbb{S}^1, 1) \to \pi_1(\mathbb{S}^1, 1)$ is the multiplication by $k$ in $\mathbb{Z}$.
(1) Consider the following map $T_a\colon \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$, called the *Dehn twist along the curve a*.




The loop $a$ is taken to itself, whereas the loop $b$ is taken to the diagonal loop $c$ pictured on the right-hand side. In general, each vertical loop on the left-hand side is skewed to the right as it travels upwards, so that it becomes one of the 45-degree diagonal loops on the right-hand side.
Describe the induced homomorphism $\pi_1(T_a)$ on the fundamental group $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1) = \mathbb{Z} \times \mathbb{Z}$.

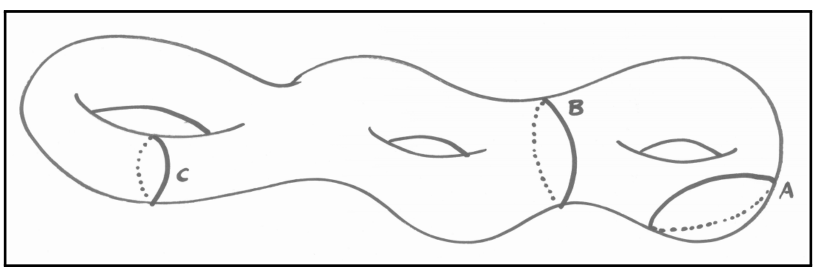
(2) Dehn twists may be defined more generally for surfaces. Given a piece of a surface, homeomorphic to a cylinder, one may define the Dehn twist $T_a$ along $a$ as follows:




(it acts by the identity outside of the shaded region). Taking $a$ to be one of the standard generators for the fundamental group of $F_2$ (recall this from lectures), describe the induced homomorphism $\pi_1(T_a)\colon \pi_1(F_2) \to \pi_1(F_2)$.

A result that will soon appear in lectures is the fact that the first homology $H_1(X)$ of a path-connected space $X$ is isomorphic to the abelianisation of its fundamental group $\pi_1(X, x)$ based at any point $x \in X$.
(3)* Using the computations of the fundamental groups of orientable and non-orientable surfaces from the lecture, compute their first homology groups.
(4)* Let $F_{g,n}$ be the orientable surface of genus $g$ with $n > 0$ points removed. Compute its fundamental group und its first homology group. (Hint: Write $F_{g,n}$ "in normal form", that means as a quotient space of a regular $4g$-gon; draw $n-1$ extra (not necessarily straight) edges from one corner to another or the same corner; now remove in each of the $n$ "compartments" one interior point; find a retraction onto the subspace which consists of the $4g$ edges on the boundary and the $n-1$ extra edges.)

**Exercise 6.3** (Nullhomotopies and nullhomologies.)
Consider the following three curves on the surface $F_3$.



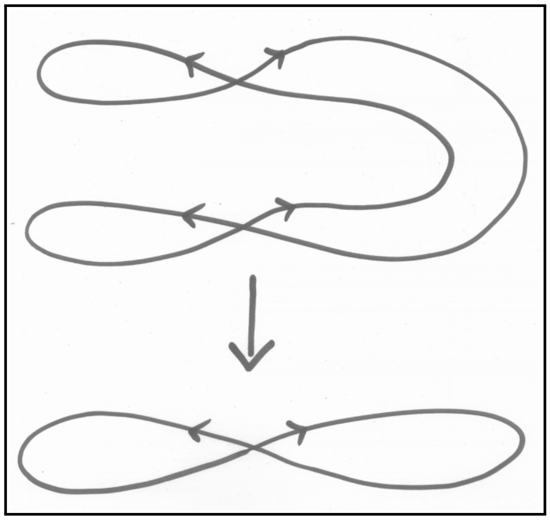

(a) Observe that the curve $A$ is nullhomotopic.
(b) Construct a 2-chain whose boundary is equal to the 1-cycle represented by the curve $B$. Thus, $B$ is nullhomologous. (Write $F_3$ in normal form as above and use the obvious triangulation.)
(c)* However, $B$ is not nullhomotopic (show this using your knowledge of $\pi_1(F_3)$ ; this is harder than one expects).
(d)* Show that the curve $C$ is neither nullhomotopic nor nullhomologous. (Consider the commutator subgroup of $\pi_1(F_3)$, which also gives an alternative way to deduce that $B$ is nullhomologous.)

**Exercise 6.4** (Disjoint unions of spaces.)
Let $X$ be a topological space which splits as the topological disjoint union of subspaces $X = \bigsqcup_\alpha X_\alpha$. Show that the singular chain complex $S_\bullet(X)$ of $X$ splits into a direct sum of summands indexed by $\alpha$, and that the boundary operator $\partial$ preserves the summands. Deduce that the subcomplexes of cycles and of boundaries also split with

respect to $\alpha$, and therefore so does the homology of $X$, in other words we have, for each $n$,

$$H_n\Big(\bigsqcup_\alpha X_\alpha\Big) \cong \bigoplus_\alpha H_n(X_\alpha).$$

**Exercise 6.5** (Coverings and $H_1$.)
Let $\xi\colon \tilde{X} \to X$ be a covering. Recall from the lecture that the map of fundamental groups $\pi_1(\xi)\colon \pi_1(\tilde{X}, \tilde{x}) \to \pi_1(X, x)$ is injective. Consider the covering



of $X = \mathbb{S}^1 \vee \mathbb{S}^1$.
(a) Show that $H_1(\tilde{X}) \cong \mathbb{Z}^3$, whereas $H_1(X) \cong \mathbb{Z}^2$.
(b) Compute the homomorphism $\pi_1(\xi)$ of fundamental groups induced by $\xi$, then abelianise this to compute the homomorphism $H_1(\xi)$ that it induces on first homology. Deduce that coverings do not always induce injective maps on homology.

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**Exercise 6.6*** (Multi-valued functions: integrating on non-simply-connected domains.)
Let $\Omega \subset \mathbb{C}$ be a region (i.e., open and connected) and $z_0 \in \Omega$, and consider a holomorphic function $f\colon \Omega \to \mathbb{C}$; we assume that $f'(z) \neq 0$ for all $z \in \Omega$.
We would like to define a new function

$$z \mapsto \int_w f(\zeta)d\zeta \;:=\; \int_0^1 f(w(t))\dot{w}(t)dt,$$

where $w$ is a path in $\Omega$ from $z_0$ to $z$; but this path integral depends on the path $w$ and not just on its endpoint $w(1) = z$; so we would get a multi-valued function. However, — since $f$ is holomorphic —, it depends only on the homotopy class $[w]$, not on the actual path. This is our chance: If $\xi\colon \tilde{\Omega} \to \Omega$ denotes the universal covering of $\Omega$, we define a function

$$\tilde{F}\colon \tilde{\Omega} \to \mathbb{C}, \quad \tilde{F}([w], z) := \int_w f(\zeta)d\zeta = \int_0^1 f(w(t))\dot{w}(t)dt.$$

(1) $\tilde{F}$ is well-defined.
(2) $\tilde{F}$ is holomorphic. (N.B: $\tilde{\Omega}$ is a holomorphic manifold, or a Riemann surface; cf. Exercise 3.2.)
(3) Now define the *period homomorphism* $\mathrm{Per}_f\colon \pi_1(\Omega, z_0) \to \mathbb{C}$ as follows:

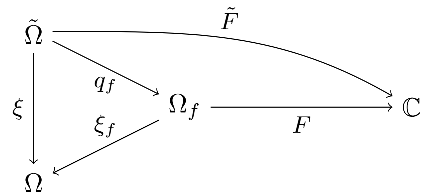
$$\mathrm{Per}_f([w]) = \int_w f(\zeta)d\zeta.$$

Convince yourself of the formulae:
$\mathrm{Per}_f(\alpha\beta) = \mathrm{Per}_f(\alpha) + \mathrm{Per}_f(\beta)$, $\mathrm{Per}_f(\alpha^{-1}) = -\mathrm{Per}_f(\alpha)$, $\mathrm{Per}_f(1) = 0$, which say that $\mathrm{Per}_f$ is a homomorphism.
There are more formulae like:
$\mathrm{Per}_{f+g}(\alpha) = \mathrm{Per}_f(\alpha) + \mathrm{Per}_g(\alpha)$, $\mathrm{Per}_{\lambda f}(\alpha) = \lambda\,\mathrm{Per}_f(\alpha)$, $\mathrm{Per}_{\overline{f}}(\alpha) = \overline{\mathrm{Per}_f(\alpha)}$, which say what ?

Next conclude, that the kernel $K := \ker(\mathrm{Per}_f) \leqslant \pi_1(\Omega, z_0)$ of $\mathrm{Per}_f$ contains at least the commutator subgroup of $\pi_1(\Omega, z_0)$. Now let $\xi_f \colon \Omega_f \to \Omega$ be the covering corresponding to that subgroup $K$, i.e., the quotient of $\tilde{\Omega}$ by the action of $K$ by deck transformations. Denote this quotient map by $q_f$.
Show that

$$\tilde{F}([a * w], z) = \tilde{F}([w], z) + \mathrm{Per}_f([a]),$$

where $a$ is a closed loop based at $z_0$ and $w$ is any path from $z_0$ to $z$. Conclude that $\tilde{F}$ factors as the composite of $q_f$ followed by a well-defined map $F \colon \Omega_f \to \mathbb{C}$. Summarising, we have the diagram:




Thus we have found the natural domain of (*well*-)definition of the multi-valued function $z \mapsto \int_{z_0}^{z} f$.

(5) Examples.
In each example, describe $\pi_1(\Omega, z_0)$, compute the period homomorphism and describe the covering $\xi_f \colon \Omega_f \to \Omega$ and the function $F$.
(5.1) : Take $\Omega = \mathbb{C} - \{0\}$ and $f(z) = \frac{1}{z}$.
(5.2) : Take $\Omega = \mathbb{C} - \{-1, 1\}$ and $f(z) = \frac{1}{1+z} + \frac{1}{1-z}$.
(5.3) : Take $\Omega = \mathbb{C} - \{-1, 1\}$ and $f(z) = \frac{a}{1+z} + \frac{b}{1-z}$, for integers $a, b \in \mathbb{Z}$.
(5.4) : Take $\Omega = \mathbb{C} - \{-1, 1\}$ and $f(z) = \frac{1}{1+z} + \frac{\pi}{1-z}$.
In the last three examples, feel free to build a model of the covering $\Omega_f$ as demonstrated in lectures.

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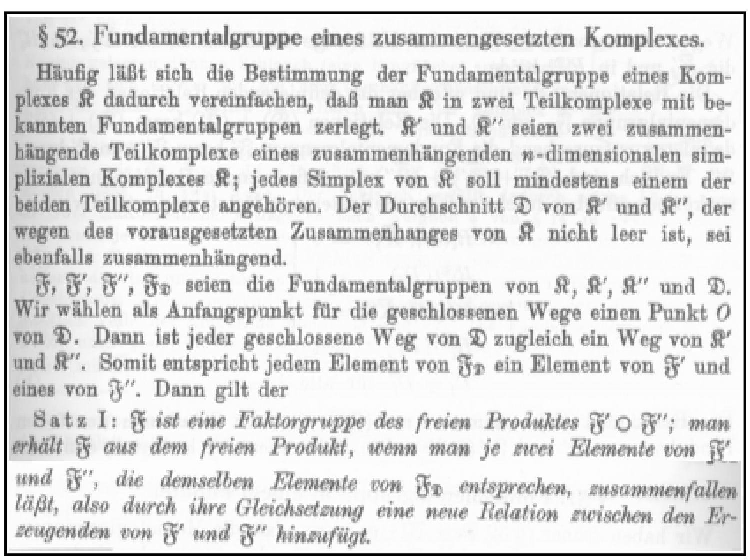

**§ 52. Fundamentalgruppe eines zusammengesetzten Komplexes.**

Häufig läßt sich die Bestimmung der Fundamentalgruppe eines Komplexes $\mathfrak{K}$ dadurch vereinfachen, daß man $\mathfrak{K}$ in zwei Teilkomplexe mit bekannten Fundamentalgruppen zerlegt. $\mathfrak{K}'$ und $\mathfrak{K}''$ seien zwei zusammenhängende Teilkomplexe eines zusammenhängenden $n$-dimensionalen simplizialen Komplexes $\mathfrak{K}$; jedes Simplex von $\mathfrak{K}$ soll mindestens einem der beiden Teilkomplexe angehören. Der Durchschnitt $\mathfrak{D}$ von $\mathfrak{K}'$ und $\mathfrak{K}''$, der wegen des vorausgesetzten Zusammenhanges von $\mathfrak{K}$ nicht leer ist, sei ebenfalls zusammenhängend.

$\mathfrak{F}$, $\mathfrak{F}'$, $\mathfrak{F}''$, $\mathfrak{F}_{\mathfrak{D}}$ seien die Fundamentalgruppen von $\mathfrak{K}$, $\mathfrak{K}'$, $\mathfrak{K}''$ und $\mathfrak{D}$. Wir wählen als Anfangspunkt für die geschlossenen Wege einen Punkt $O$ von $\mathfrak{D}$. Dann ist jeder geschlossene Weg von $\mathfrak{D}$ zugleich ein Weg von $\mathfrak{K}'$ und $\mathfrak{K}''$. Somit entspricht jedem Element von $\mathfrak{F}_{\mathfrak{D}}$ ein Element von $\mathfrak{F}'$ und eines von $\mathfrak{F}''$. Dann gilt der

Satz I: *$\mathfrak{F}$ ist eine Faktorgruppe des freien Produktes $\mathfrak{F}' \circ \mathfrak{F}''$; man erhält $\mathfrak{F}$ aus dem freien Produkt, wenn man je zwei Elemente von $\mathfrak{F}'$ und $\mathfrak{F}''$, die demselben Elemente von $\mathfrak{F}_{\mathfrak{D}}$ entsprechen, zusammenfallen läßt, also durch ihre Gleichsetzung eine neue Relation zwischen den Erzeugenden von $\mathfrak{F}'$ und $\mathfrak{F}''$ hinzufügt.*

The Seifert-van-Kampen Theorem, from *Lehrbuch der Topologie*, H. Seifert and W. Threlfall.