Aufgaben zur Topologie

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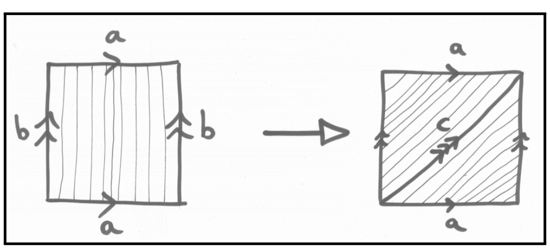
Week 6 — Fundamental groups and first homology groups Due: 7. December 2016

Exercise 6.1 (Gradients, rotation and divergence: grad, rot and div.)
(1) Let X be an open subset of $\mathbb{R}^2$ and define real vector spaces as follows.
◦ $C_0 = C^\infty(X)$, the space of smooth real-valued functions on X.
◦ $C_{-1} = C^\infty(X) \times C^\infty(X)$, to be thought of as the space of smooth vector fields $v = (v_1, v_2)$ on X, in coordinates.
◦ $C_{-2} = C^\infty(X)$, to be thought of as the space of volume forms on X.
There are linear maps $\mathrm{grad}\colon C_0 \to C_{-1}$ and $\mathrm{rot}\colon C_{-1} \to C_{-2}$ defined by $\mathrm{grad}(f) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ and $\mathrm{rot}(v_1, v_2) = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}$.
(a) Show (using vector calculus) that this is a chain complex (over the field $\mathbb{R}$), where all undefined chain modules are 0.
(b) For arbitrary X, find the dimension of $H_0(C_\bullet)$, i.e., the kernel of grad.
(c) When X is not simply-connected, give an example of a vector field that has zero rotation, but is not the gradient of any smooth function on X, thus showing that the homology group $H_{-1}(C_\bullet)$ is non-trivial in this case.

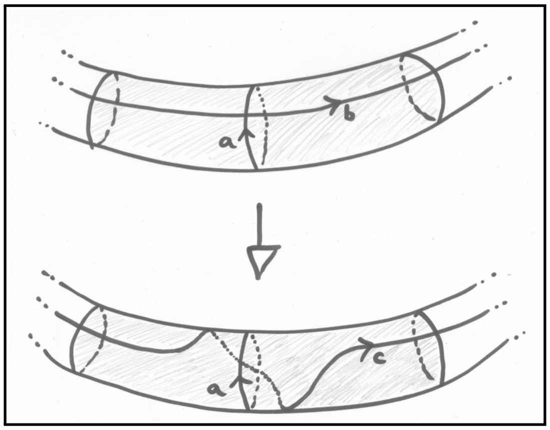
(2) Now let X be an open subset of $\mathbb{R}^3$ and define real vector spaces as follows.
◦ $C_0 = C^\infty(X)$.
◦ $C_{-1} = C^\infty(X) \times C^\infty(X) \times C^\infty(X)$.
◦ $C_{-2} = C^\infty(X) \times C^\infty(X) \times C^\infty(X)$.
◦ $C_{-3} = C^\infty(X)$.
(a) Recall the definitions of the linear operators $\mathrm{grad}\colon C_0 \to C_{-1}$, $\mathrm{rot}\colon C_{-1} \to C_{-2}$ and $\mathrm{div}\colon C_{-2} \to C_{-3}$ in this setting, and show that these form a chain complex.
(b) As above, compute the dimension of $H_0(C_\bullet)$.
(c) Show that the homology group $H_{-1}(C_\bullet)$ is non-trivial when X is not simply-connected by finding a vector field with zero rotation and which is not the gradient of any smooth function on X.
(d)* Find an X such that $H_{-2}(C_\bullet)$ is non-trivial, i.e., we need a vector field defined on X with zero divergence and which is not the rotation of any other vector field on X. (A first case to consider is $X = \mathbb{R}^3 - \{(0,0,0)\}$.)

Exercise 6.2 (Induced maps on π_1 and the abelianisation of π_1.)
Recall: If $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ is a map of degree k, then the induced map $\pi_1(f)\colon \pi_1(\mathbb{S}^1, 1) \to \pi_1(\mathbb{S}^1, 1)$ is the multiplication by k in $\mathbb{Z}$.
(1) Consider the following map $T_a\colon \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$, called the *Dehn twist along the curve a*.



The loop a is taken to itself, whereas the loop b is taken to the diagonal loop c pictured on the right-hand side. In general, each vertical loop on the left-hand side is skewed to the right as it travels upwards, so that it becomes one of the 45-degree diagonal loops on the right-hand side.
Describe the induced homomorphism $\pi_1(T_a)$ on the fundamental group $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1) = \mathbb{Z} \times \mathbb{Z}$.

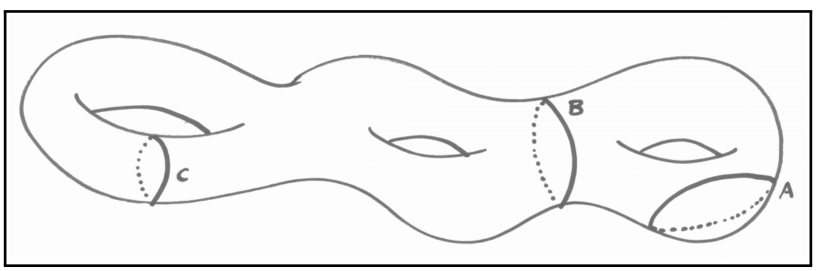
(2) Dehn twists may be defined more generally for surfaces. Given a piece of a surface, homeomorphic to a cylinder, one may define the Dehn twist T_a along a as follows:



(it acts by the identity outside of the shaded region). Taking a to be one of the standard generators for the fundamental group of F_2 (recall this from lectures), describe the induced homomorphism $\pi_1(T_a)\colon \pi_1(F_2) \to \pi_1(F_2)$.

A result that will soon appear in lectures is the fact that the first homology $H_1(X)$ of a path-connected space X is isomorphic to the abelianisation of its fundamental group $\pi_1(X, x)$ based at any point $x \in X$.
(3)* Using the computations of the fundamental groups of orientable and non-orientable surfaces from the lecture, compute their first homology groups.
(4)* Let $F_{g,n}$ be the orientable surface of genus g with $n > 0$ points removed. Compute its fundamental group und its first homology group. (Hint: Write $F_{g,n}$ "in normal form", that means as a quotient space of a regular $4g$-gon; draw $n-1$ extra (not necessarily straight) edges from one corner to another or the same corner; now remove in each of the n "compartments" one interior point; find a retraction onto the subspace which consists of the $4g$ edges on the boundary and the $n-1$ extra edges.)

Exercise 6.3 (Nullhomotopies and nullhomologies.)
Consider the following three curves on the surface F_3.



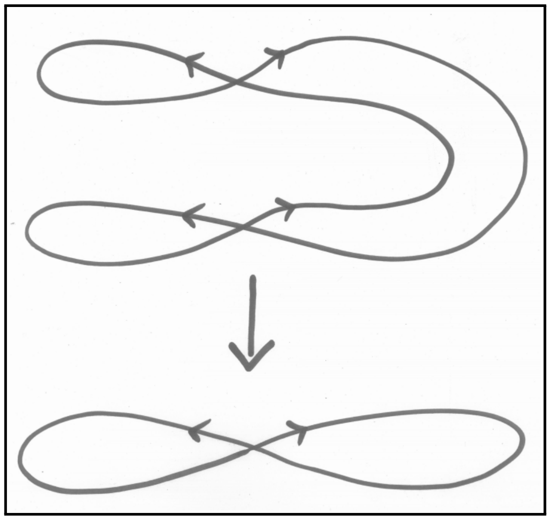
(a) Observe that the curve A is nullhomotopic.
(b) Construct a 2-chain whose boundary is equal to the 1-cycle represented by the curve B. Thus, B is nullhomologous. (Write F_3 in normal form as above and use the obvious triangulation.)
(c)* However, B is not nullhomotopic (show this using your knowledge of $\pi_1(F_3)$; this is harder than one expects).
(d)* Show that the curve C is neither nullhomotopic nor nullhomologous. (Consider the commutator subgroup of $\pi_1(F_3)$, which also gives an alternative way to deduce that B is nullhomologous.)

Exercise 6.4 (Disjoint unions of spaces.)
Let X be a topological space which splits as the topological disjoint union of subspaces $X = \bigsqcup_\alpha X_\alpha$. Show that the singular chain complex $S_\bullet(X)$ of X splits into a direct sum of summands indexed by α, and that the boundary operator ∂ preserves the summands. Deduce that the subcomplexes of cycles and of boundaries also split with

respect to α, and therefore so does the homology of X, in other words we have, for each n,

$$H_n\Big(\bigsqcup_\alpha X_\alpha\Big) \;\cong\; \bigoplus_\alpha H_n(X_\alpha).$$

Exercise 6.5 (Coverings and H_1.)
Let $\xi\colon \tilde{X} \to X$ be a covering. Recall from the lecture that the map of fundamental groups $\pi_1(\xi)\colon \pi_1(\tilde{X}, \tilde{x}) \to \pi_1(X, x)$ is injective. Consider the covering



of $X = \mathbb{S}^1 \vee \mathbb{S}^1$.
(a) Show that $H_1(\tilde{X}) \cong \mathbb{Z}^3$, whereas $H_1(X) \cong \mathbb{Z}^2$.
(b) Compute the homomorphism $\pi_1(\xi)$ of fundamental groups induced by ξ, then abelianise this to compute the homomorphism $H_1(\xi)$ that it induces on first homology. Deduce that coverings do not always induce injective maps on homology.

Exercise 6.6* (Multi-valued functions: integrating on non-simply-connected domains.)
Let $\Omega \subset \mathbb{C}$ be a region (i.e., open and connected) and $z_0 \in \Omega$, and consider a holomorphic function $f\colon \Omega \to \mathbb{C}$; we assume that $f'(z) \neq 0$ for all $z \in \Omega$.
We would like to define a new function

$$z \mapsto \int_w f(\zeta) d\zeta \;:=\; \int_0^1 f(w(t))\dot{w}(t)dt,$$

where w is a path in Ω from z_0 to z; but this path integral depends on the path w and not just on its endpoint $w(1) = z$; so we would get a multi-valued function. However, — since f is holomorphic —, it depends only on the homotopy class $[w]$, not on the actual path. This is our chance: If $\xi\colon \tilde{\Omega} \to \Omega$ denotes the universal covering of Ω, we define a function

$$\tilde{F}\colon \tilde{\Omega} \to \mathbb{C}, \quad \tilde{F}([w], z) := \int_w f(\zeta) d\zeta = \int_0^1 f(w(t))\dot{w}(t)dt.$$

(1) $\tilde{F}$ is well-defined.
(2) $\tilde{F}$ is holomorphic. (N.B: $\tilde{\Omega}$ is a holomorphic manifold, or a Riemann surface; cf. Exercise 3.2.)
(3) Now define the *period homomorphism* $\mathrm{Per}_f\colon \pi_1(\Omega, z_0) \to \mathbb{C}$ as follows:

$$\mathrm{Per}_f([w]) = \int_w f(\zeta) d\zeta.$$

Convince yourself of the formulae:
$\mathrm{Per}_f(\alpha\beta) = \mathrm{Per}_f(\alpha) + \mathrm{Per}_f(\beta)$, $\mathrm{Per}_f(\alpha^{-1}) = -\mathrm{Per}_f(\alpha)$, $\mathrm{Per}_f(1) = 0$, which say that Per_f is a homomorphism.
There are more formulae like:
$\mathrm{Per}_{f+g}(\alpha) = \mathrm{Per}_f(\alpha) + \mathrm{Per}_g(\beta)$, $\mathrm{Per}_{\lambda f}(\alpha) = \lambda\,\mathrm{Per}_f(\alpha)$, $\mathrm{Per}_{\overline{f}}(\alpha) = \overline{\mathrm{Per}_f(\alpha)}$, which say what ?

Next conclude, that the kernel $K := \ker(\mathrm{Per}_f) \leqslant \pi_1(\Omega, z_0)$ of Per_f contains at least the commutator subgroup of $\pi_1(\Omega, z_0)$. Now let $\xi_f \colon \Omega_f \to \Omega$ be the covering corresponding to that subgroup K, i.e., the quotient of $\tilde{\Omega}$ by the action of K by deck transformations. Denote this quotient map by q_f.
Show that

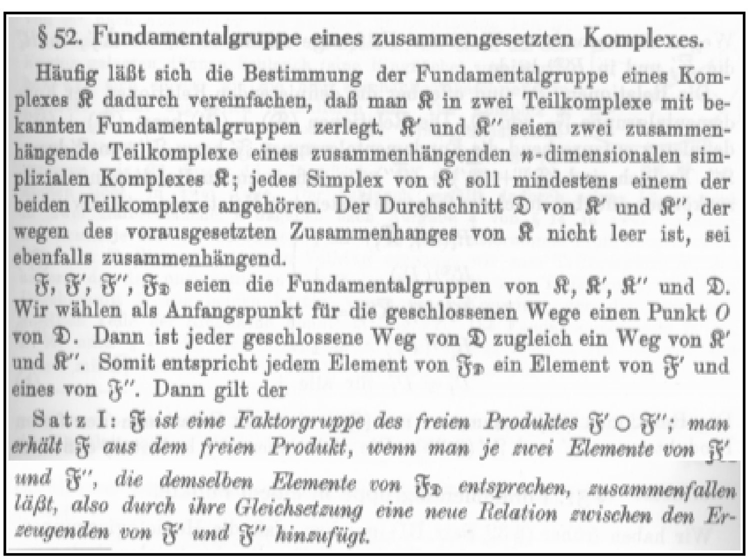
$$\tilde{F}([a * w], z) = \tilde{F}([w], z) + \mathrm{Per}_f([a]),$$

where a is a closed loop based at z_0 and w is any path from z_0 to z. Conclude that $\tilde{F}$ factors as the composite of q_f followed by a well-defined map $F \colon \Omega_f \to \mathbb{C}$. Summarising, we have the diagram:

$$\begin{tikzcd}
\tilde{\Omega} \arrow[dr, "q_f"] \arrow[rr, bend left, "\tilde{F}"] \arrow[dd, "\xi"'] & & \\
& \Omega_f \arrow[r, "F"'] \arrow[dl, "\xi_f"'] & \mathbb{C} \\
\Omega & &
\end{tikzcd}$$

Thus we have found the natural domain of (*well*-)definition of the multi-valued function $z \mapsto \int_{z_0}^{z} f$.

(5) Examples.
In each example, describe $\pi_1(\Omega, z_0)$, compute the period homomorphism and describe the covering $\xi_f \colon \Omega_f \to \Omega$ and the function F.
(5.1) : Take $\Omega = \mathbb{C} - \{0\}$ and $f(z) = \frac{1}{z}$.
(5.2) : Take $\Omega = \mathbb{C} - \{-1, 1\}$ and $f(z) = \frac{1}{1+z} + \frac{1}{1-z}$.
(5.3) : Take $\Omega = \mathbb{C} - \{-1, 1\}$ and $f(z) = \frac{a}{1+z} + \frac{b}{1-z}$, for integers $a, b \in \mathbb{Z}$.
(5.4) : Take $\Omega = \mathbb{C} - \{-1, 1\}$ and $f(z) = \frac{1}{1+z} + \frac{\pi}{1-z}$.
In the last three examples, feel free to build a model of the covering Ω_f as demonstrated in lectures.



§ 52. Fundamentalgruppe eines zusammengesetzten Komplexes.

Häufig läßt sich die Bestimmung der Fundamentalgruppe eines Komplexes $\mathfrak{K}$ dadurch vereinfachen, daß man $\mathfrak{K}$ in zwei Teilkomplexe mit bekannten Fundamentalgruppen zerlegt. $\mathfrak{K}'$ und $\mathfrak{K}''$ seien zwei zusammenhängende Teilkomplexe eines zusammenhängenden n-dimensionalen simplizialen Komplexes $\mathfrak{K}$; jedes Simplex von $\mathfrak{K}$ soll mindestens einem der beiden Teilkomplexe angehören. Der Durchschnitt $\mathfrak{D}$ von $\mathfrak{K}'$ und $\mathfrak{K}''$, der wegen des vorausgesetzten Zusammenhanges von $\mathfrak{K}$ nicht leer ist, sei ebenfalls zusammenhängend.

$\mathfrak{F}$, $\mathfrak{F}'$, $\mathfrak{F}''$, $\mathfrak{F}_{\mathfrak{D}}$ seien die Fundamentalgruppen von $\mathfrak{K}$, $\mathfrak{K}'$, $\mathfrak{K}''$ und $\mathfrak{D}$. Wir wählen als Anfangspunkt für die geschlossenen Wege einen Punkt O von $\mathfrak{D}$. Dann ist jeder geschlossene Weg von $\mathfrak{D}$ zugleich ein Weg von $\mathfrak{K}'$ und $\mathfrak{K}''$. Somit entspricht jedem Element von $\mathfrak{F}_{\mathfrak{D}}$ ein Element von $\mathfrak{F}'$ und eines von $\mathfrak{F}''$. Dann gilt der

Satz I: *$\mathfrak{F}$ ist eine Faktorgruppe des freien Produktes $\mathfrak{F}' \circ \mathfrak{F}''$; man erhält $\mathfrak{F}$ aus dem freien Produkt, wenn man je zwei Elemente von $\mathfrak{F}'$ und $\mathfrak{F}''$, die demselben Elemente von $\mathfrak{F}_{\mathfrak{D}}$ entsprechen, zusammenfallen läßt, also durch ihre Gleichsetzung eine neue Relation zwischen den Erzeugenden von $\mathfrak{F}'$ und $\mathfrak{F}''$ hinzufügt.*

The Seifert-van-Kampen Theorem, from *Lehrbuch der Topologie*, H. Seifert and W. Threlfall.