Aufgaben zur Topologie

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Week 5 — Classification of coverings; Seifert-van-Kampen Theorem Due: 30. November 2016

Exercise 5.1 (Coverings and homeomorphisms.)
If $\xi\colon \tilde{X} \to X$ is a covering of a locally path-connected space X all of whose path-components are 1-connected, then $\tilde{X}$ is homeomorphic to a disjoint union of copies of X; if, in addition, $\tilde{X}$ is 0-connected, then ξ is a homeomorphism.

Exercise 5.2 (Connected coverings of Lie groups are Lie groups.)

Exercise 5.3 (Fundamental groups of topological groups are abelian.)
(1) Let G be a topological group; we denote its multiplication by $\mu = \centerdot\colon G \times G \to G, (x, y) \mapsto x \centerdot y$ and the inverse by ${}^{-1}\colon G \to G, x \mapsto x^{-1}$, and we take the neutral element 1 as basepoint. Consider for two pointed maps $a, b\colon \mathbb{S}^1 \to G$ the pointwise multiplication $(a \centerdot b)(t) := a(t) \centerdot b(t)$, and the pointwise inversion $a^{-1}(t) := a(t)^{-1}$. Show that this is a group structure on the set $M = \mathrm{maps}((\mathbb{S}^1, 1), (G, 1))$ of based maps. Convince yourself that all of this is continuous in the compact-open topology on M.
(2) Show that this group structure induces a well-defined group structure on the set of based homotopy classes, that is on $[(\mathbb{S}^1, 1), (G, 1)] = \pi_1(G, 1)$, by setting $[a] \centerdot [b] := [a \centerdot b]$ and $[a]^{-1} := [a^{-1}]$, where the homotopy class of the constant map $t \mapsto 1$ is the neutral element.
(3) Recall now the old group structure on $\pi_1(G, 1)$, denoted here by $[a] * [b] = [a * b]$ and $[a]^{-1} = [\bar{a}]$, where $a * b$ is the concatenation of two paths and $\bar{a}$ is the reverse path. Show (by pictures, not by formulae) that the multiplications satisfy the following exchange property: $([a] * [b]) \centerdot ([c] * [d]) = ([a] \centerdot [c]) * ([b] \centerdot [d])$.
(4) Assume that S is a set with two group structures $*$ and $\centerdot$ satisfying the exchange property. Then the group structures agree ($* = \centerdot$) and are abelian. (Not all of the group axioms are needed for the proof of this statement; which ones are used?)
(5) Thus the statement is proved.

Exercise 5.4 (Homotopy invariance of pull-backs.)
Let $\xi\colon \tilde{X} \to X$ be a covering and let $f_0, f_1\colon Y \to X$ be two maps. Denote the pullbacks (for $i = 0, 1$) of ξ by $\xi_i = f_i^*(\xi)\colon Y_i = f_i^*(\tilde{X}) \to Y$.
(1) If f_0 and f_1 are homotopic, there is a homeomorphism $\Phi\colon f_0^*(\tilde{X}) \to f_1^*(\tilde{X})$ with $\xi_1 \circ \Phi = \xi_0$.
(Hint: Consider the pull-back $F^*(\tilde{X}) \to Y \times [0, 1]$ of ξ along a homotopy F between f_0 and f_1, and lift the path $t \mapsto (y, t)$ with an arbitrary starting point in $f_0^*(\tilde{X}) \subset F^*(\tilde{X})$ over $(y, 0)$.)
(2) If $f\colon Y \to X$ is null-homotopic, then $f^*(\xi)$ is a trivial covering.
(3) Application: The inclusions $\iota_n\colon \mathbb{S}^1 \cong \mathbb{R}P^1 \hookrightarrow \mathbb{R}P^n$ are not null-homotopic; even better, ι_n induces isomorphisms on fundamental groups.

Exercise 5.5 (Branched coverings and polynomials.) Let $p(z) = z^n + a_{n-1}z^{n-1} + \ldots a_1 z + a_0$ be a non-constant complex polynomial. Denote by S the set of all critical points, i.e., points z with $p'(z) = 0$ and V the set of all critical values $v = p(\zeta)$ for $\zeta \in S$.
(1) Show that $p\colon \mathbb{C} - S \to \mathbb{C} - V$ is an n-fold covering.
(Hint: $\mathbb{C}$ is locally path-connected and V is a closed subset (why?), so for each $z \in \mathbb{C} - V$ you may find a connected open neighbourhood U of z in $\mathbb{C} - V$. Study the preimage $p^{-1}(U)$ and use the Inverse Function Theorem.)

Now we consider the special cases $p_n(z) = z^n$ for $n \geq 2$ as maps from the disc $B_r(0)$ of radius $r > 0$ around 0 to the disc $B_{r^n}(0)$. We just saw that the restriction $p_n|\colon B_r(0) - 0 \to B_{r^n}(0) - 0$ is an n-fold covering. But what is $p_n\colon B_r(0) \to B_{r^n}(0)$, where $\zeta = 0$ has only one point in its pre-image and not n points (as do all other points) ? The

map p_n is a prototypical example of a so-called *branched covering*; we will not define this notion in all generality, but want to prove that a non-constant complex polynomial p is a branched covering in the following sense:
(2) Show that for each critical value $v \in V$ there is a neighbourhood $U \subset \mathbb{C}$ and a partition $k_1+k_2+\cdots+k_l = n$, such that the restriction $p|: p^{-1}(U) \to U$ is homeomorphic to the Whitney sum of l branched coverings $p_{k_1}, p_{k_2}, \ldots, p_{k_l}$ as considered above.
(Note: we call two coverings $\xi\colon \tilde{X} \to X$ and $\nu\colon \tilde{Y} \to Y$ homeomorphic if there are homeomorphisms $\phi\colon X \to Y$ and $\tilde{\phi}\colon \tilde{X} \to \tilde{Y}$ such that $\nu \circ \tilde{\phi} = \phi \circ \xi$.)
(3)* What happens in a neighbourhood of ∞ if we extend p to $\mathbb{S}^2 \to \mathbb{S}^2$ by setting $p(\infty) = \infty$?

Die Schar der konzentrischen Kreise $r =$ konst. geht also in die Schar der konzentrischen Kreise $\varrho =$ konst. über, und die Winkel φ werden verdoppelt. Läßt man also φ bei festem $r > 0$ von 0 bis 2π laufen, so variiert ϑ monoton zwischen 0 und 4π, und der Bildpunkt ζ durchwandert den Kreis $\varrho =$ konst. zweimal, wenn der Punkt z den Kreis $r =$ konst. einmal durchläuft. Somit entsprechen einem Werte von ζ zwei Werte von z; d. h. die Umkehrfunktion

$$z = \sqrt{\zeta}$$

unserer Funktion $\zeta = z^2$ ist eine „zweideutige" Funktion von ζ.

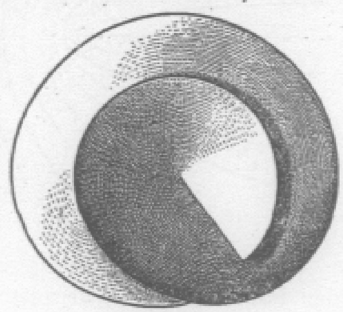


Abb. 95.

Zur Veranschaulichung dieser Verhältnisse denkt man sich die ζ-Ebene in zwei Exemplaren, zwei „*Blättern*", übereinandergelegt. Dann führt man in beiden Blättern von $\zeta = 0$ bis $\zeta = \infty$ kongruente Schnitte, etwa längs der positiven reellen Achse, und heftet die vier entstehenden Schnittufer unter wechselweiser Durchdringung der beiden Blätter kreuzweise aneinander (Abb. 95).

A discussion of the function $z \mapsto z^2$ and its inverse $\zeta \mapsto \sqrt{\zeta}$ from *Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen* by A. Hurwitz and R. Courant, using polar coordinates $z = re^{i\varphi}$ and $\zeta = z^2 = \varrho e^{i\vartheta}$.

Exercise 5.6* (Spaces with fundamental group $\mathbb{Z}/n\mathbb{Z}$.)
Let $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ be a map of degree $\mathrm{grad}(f) = n$. Consider the space $M(n) = \mathbb{S}^1 \cup_f \mathbb{D}^2$ obtained by attaching a 2-disc to a circle along its boundary using the map f — i.e., the quotient $(\mathbb{S}^1 \sqcup \mathbb{D}^2)/\sim$ where $\sim$ is the equivalence relation generated by the relations $\zeta \sim f(\zeta)$ for $\zeta \in \partial\mathbb{D}^2 = \mathbb{S}^1$.
(1) Make a sketch of this identification.
(2) Show that $\pi_1(M(n)) \cong \mathbb{Z}/n\mathbb{Z}$.

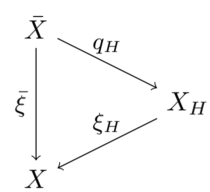
Exercise 5.7* (Any group is the fundamental group of some space.)
(1) Let G be a group with finite presentation $\langle s_1, \ldots, s_n \mid r_1, \ldots, r_k \rangle$. Using a similar idea to Exercise 5.6, and the Seifert-van-Kampen Theorem multiple times, construct a space X such that $\pi_1(X) \cong G$. First find a space Y_0 whose fundamental group is the free group $\langle s_1, \ldots, s_n \mid \rangle$, then attach a 2-disc to form a space Y_1 with fundamental group $\langle s_1, \ldots, s_n \mid r_1 \rangle$, and so on, until you find $Y_k = X$.
(2) Now suppose that G is any group, not necessarily possessing a finite presentation, or even a finite generating set (think of $G = \mathbb{Q}$, for example, or $G = S^1$, considered as an abstract (uncountable!) group). Using a limit argument, show that there is nevertheless a space X with fundamental group G. You may use the following facts:
(a) Suppose that X is path-connected and is the union of a family of path-connected open subspaces X_α. Assume that each intersection $X_\alpha \cap X_\beta$ is X_γ for some γ. Also assume that X and each X_α are "nice" (i.e., locally 0-connected and semi-locally 1-connected). Let $x \in \bigcap_\alpha X_\alpha$. Then $\pi_1(X, x)$ is the direct limit of the subgroups $\pi_1(X_\alpha, x)$.
(b) Any group is the direct limit of the family of all of its finitely presentable subgroups.

Note: There is a subtlety with the limit argument if one tries to use fact (b): it is not possible to pick a finite presentation for each finitely present*able* subgroup of G in a way that is compatible with all inclusions between them. To rectify this, you can instead use the following modification of fact (b):
(c) Any group G is the direct limit of the diagram of groups whose objects are all finitely presentable subgroups of G equipped with a choice of presentation, and whose morphisms are just those inclusions $\langle s_1, \ldots, s_n \mid r_1, \ldots, r_k \rangle \hookrightarrow \langle s'_1, \ldots, s'_m \mid r'_1, \ldots, r'_l \rangle$ for which $\{s_1, \ldots, s_n\}$ is a subset of $\{s'_1, \ldots, s'_m\}$ and $\{r_1, \ldots, r_k\}$ is a subset of $\{r_1, \ldots, r'_l\}$.

Exercise 5.8* (Addendum to the classification theorem for coverings.)
Let X be a 0-connected, locally 0-connected and semi-locally 1-connected space. We denote by

$$\mathrm{char}\colon \mathcal{C}ov^0(X, x_0) \longrightarrow \mathcal{G}(\pi_1(X, x_0))$$

the function, which associates to an isomorphism class $[\xi] = [\xi\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)]$ of based and connected coverings the characteristic subgroup $\mathrm{char}[\xi] = \xi_*(\pi_1(X, x_0))$ of $\pi_1(X, x_0)$.
(1) $\mathrm{char}[\xi_1] \leq \mathrm{char}[\xi_2] \iff$ There is a unique morphism $\xi_1 \to \xi_2$.
(2) $\mathrm{char}[\xi]$ is normal $\iff \xi$ is regular (i.e., “fibre-transitive”).
(3) Let $H \leq \pi_1(X, x_0)$ be a subgroup and denote by $\bar{\xi}\colon \bar{X} \to X$ the universal covering of X. In the commutative diagram



we have:
(3.1) $q_H\colon \bar{X} \to X_H := \bar{X}/H$ is a universal covering; thus $\mathcal{D}(q_H) \cong H$ for the group of deck transformations.
(3.2) $\mathcal{D}(\xi_H) \cong \mathrm{Weyl}(H) = N_G(H)/H$, the Weyl group of H in $G = \pi_1(X, x_0)$.
(3.3) In particular, $\mathcal{D}(\xi_H) \cong \pi_1(X, x_0)/H$ if H is normal.