Aufgaben zur Topologie

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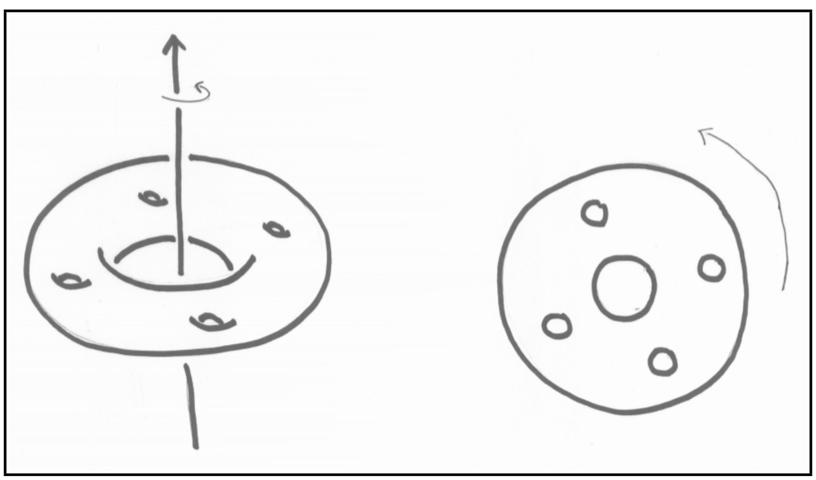
Week 4 — Coverings and deck transformations Due: 23.11.2016 (before the lecture)

Exercise 4.1 (Fundamental group of the Klein bottle)
Consider $\tilde{X} = \mathbb{R}^2$ with the translation $A\colon (x, y) \mapsto (x + 1, y)$ and the glide reflection $B\colon (x, y) \mapsto (-x, y + 1)$. Let G denote the subgroup of affine motions of $\mathbb{R}^2$ generated by A and B.
(1) This action is free and properly discontinuous.
(2) A and B satisfy the relation $ABA = B$.
(3) G is isomorphic to the semi-direct product $\mathbb{Z} \rtimes \mathbb{Z}$, where $b \in \mathbb{Z}$ acts on $\mathbb{Z}$ via $b \centerdot a := (-1)^b\, a$.
(4) The quotient $\xi\colon \tilde{X} \to X := \tilde{X}/G$ is the Klein bottle.
(5) Thus $\pi_1(K) \cong \mathbb{Z} \rtimes \mathbb{Z}$.

Exercise 4.2 (Connectivity and sections)
If $\xi_1\colon \tilde{X}_1 \to X$ and $\xi_2\colon \tilde{X}_2 \to X$ are two coverings of X, we denote by $\xi_1 \oplus \xi_2\colon \tilde{X}_1 \sqcup \tilde{X}_2 \to X$ their *sum over* X or *Whitney sum*. Show that a covering $\xi\colon \tilde{X} \to X$ has a section if and only if it is isomorphic to the sum $\xi' \oplus \mathrm{id}_X$ of some covering ξ' and the identity. Conclude that a connected covering with $k > 1$ leafs has no section. (Example: $\mathbb{S}^n \to \mathbb{R}\mathrm{P}^n$.)



The surface $F_5 \subset \mathbb{R}^3$ from Exercise 4.3(4c). The right-hand side is a bird's-eye view from above.

Exercise 4.3 (Change of fibres)
Let the discrete group G act (from the right) freely and properly discontinuously on the Hausdorff space $\tilde{X}$; thus we have a covering $\xi\colon \tilde{X} \to X := \tilde{X}/G$ with fibre G. Now assume that G also acts (from the left) on the discrete space F, not necessarily freely. We define on the space $\tilde{Y} = \tilde{X} \times F$ the "diagonal" action $g \centerdot (x, u) := (x \centerdot g^{-1}, g \centerdot u)$, and define Y to be the quotient space $\tilde{Y}/G$, which may also be denoted $\tilde{X} \times_G F$.
(1) The projection $\tilde{Y} \to \tilde{X}$ is G-equivariant, so we have a well-defined induced map $\xi_F\colon Y \to X$.
(2) This induced map is a covering with fibre F.
(3) If $F = G$ with the G-action given by left-multiplication, then $Y = \tilde{X}$ and $\xi_F = \xi$.

(4) Examples:

(a) If $\tilde{X} = \mathbb{S}^2$ with $G = \mathbb{Z}/2\mathbb{Z}$ acting by the antipodal map, then $\tilde{X} \to X$ is the two-fold covering $\mathbb{S}^2 \to \mathbb{RP}^2$. Now take $F = \mathbb{Z}/4\mathbb{Z}$ with the generator of G acting by multiplication by -1. Show that $Y \to X$ is the Whitney sum of two copies of the identity $\mathbb{RP}^2 \to \mathbb{RP}^2$ and one copy of $\mathbb{S}^2 \to \mathbb{RP}^2$. What about if $F = \mathbb{Z}/n\mathbb{Z}$ for other positive integers n, with G again acting by multiplication by -1?

(b) Let $\tilde{X} = \mathbb{R}$ with $G = \mathbb{Z}$ acting by translation; $F = \{e^{2\pi i \frac{k}{n}} \mid k = 0, \ldots, n-1\}$ with $1 \in \mathbb{Z}$ acting by rotation by $\frac{360}{n}$ degrees. What is the covering $Y \to X$ in this case?

(c) If we imagine the surface F_{g+1} embedded into $\mathbb{R}^3$ in a symmetrical way, with one large hole and g smaller holes arranged around it at equal intervals, and with the z-axis running through the large hole – see the figure on the previous page for the case $g = 4$ – then there is an action of the cyclic group $\mathbb{Z}/g\mathbb{Z}$ on it given by rotation in the (x, y)-plane by $\frac{360}{g}$ degrees. This action is free and properly discontinuous and the quotient is homeomorphic to F_2. Denote the resulting covering by $\varphi_{g+1} \colon F_{g+1} \to F_2$.

Now take $g = 4$, let $\tilde{X} = F_5$ and let $G = \mathbb{Z}/4\mathbb{Z}$ acting as described above. So $\tilde{X} \to X$ is the covering $\varphi_5 \colon F_5 \to F_2$. Let F be the set $\{-1, 1\}$ with the action of $G = \mathbb{Z}/4\mathbb{Z}$ where a generator acts by multiplication by -1. Show that the new covering $Y \to X$ is $\varphi_3 \colon F_3 \to F_2$.

(d)* More generally, if m and n are positive integers, let $\tilde{X} = F_{mn+1}$ with $G = \mathbb{Z}/mn\mathbb{Z}$ acting as described above. So $\tilde{X} \to X$ is φ_{mn+1}. Let $F = \{e^{2\pi i \frac{k}{n}} \mid k = 0, \ldots, n-1\}$ with the action of $G = \mathbb{Z}/mn\mathbb{Z}$ in which 1 acts by rotation by $\frac{360}{n}$ degrees. Then $Y \to X$ is φ_{n+1}.

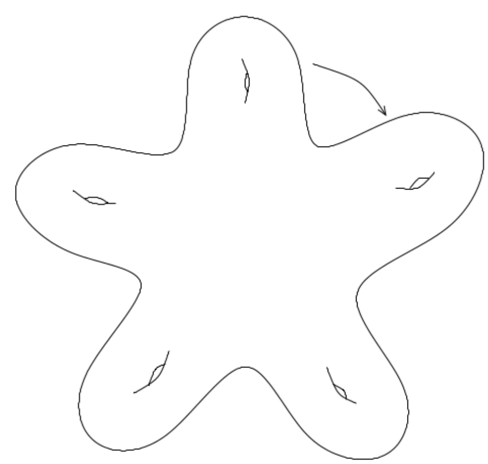


Figure 2.1 An order 5 element of $\mathrm{Mod}(S_5)$.

The surface F_5 has a 5-fold symmetry, as well as the 4-fold symmetry used in Exercise 4.3(4c).
Image credit: *A Primer on Mapping Class Groups*, B. Farb, D. Margalit, page 46.

Exercise 4.4 (Universal coverings of closed, orientable surfaces)

(a) Show that the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ given by addition in each coordinate is free and properly discontinuous. Deduce that the universal cover of the torus $\mathbb{S}^1 \times \mathbb{S}^1$ is contractible. (See part (a) of Exercise 3.3.)

(b) Consider the diagram in the hyperbolic plane $\mathbb{H}^2$ on the next page, which is to be visualised using the Poincaré disc model. The tiling is constructed as follows. Start with an octagon, centred at the middle of the disc, with 90-degree internal angles. Reflecting in L and then in M as shown on the right-hand side of the figure results in another octagon, incident to the original octagon along one edge. The composite of reflection in L and then in M is a hyperbolic isometry. Continuing in this way indefinitely, using all possible hyperbolic isometries of this form (using the analogues of the lines L and M in the smaller octagons) produces the full tiling on the left-hand side of

the figure. Let G be the group of hyperbolic isometries generated by the ones that we used to construct the tiling. Then G acts properly discontinuously and freely on $\mathbb{H}^2$.
Show that each orbit of the G-action intersects the (closed) octagon in the centre of the picture. Moreover, show that each orbit either (i) intersects the octagon exactly once in its *interior*, (ii) intersects the octagon exactly twice on its *boundary* or (iii) intersects the octagon exactly eight times on its boundary. In the latter two cases, explain how the points of intersection are related. Deduce that the universal cover of the surface F_2 is contractible, where F_g denotes the closed, orientable surface of genus g.
(c) Sketch how to adapt this proof to show that the universal cover of F_g is contractible for all $g \geqslant 1$.
(d) An application (continuation of Exercise 3.5 from last week). Let X be a path-connected (locally path-connected) space with basepoint x_0 such that $\pi_1(X, x_0)$ is finite. Show that any continuous map $X \to F_g$ must be nullhomotopic. (You may assume the fact that every element of $\pi_1(F_g)$ has infinite order.)

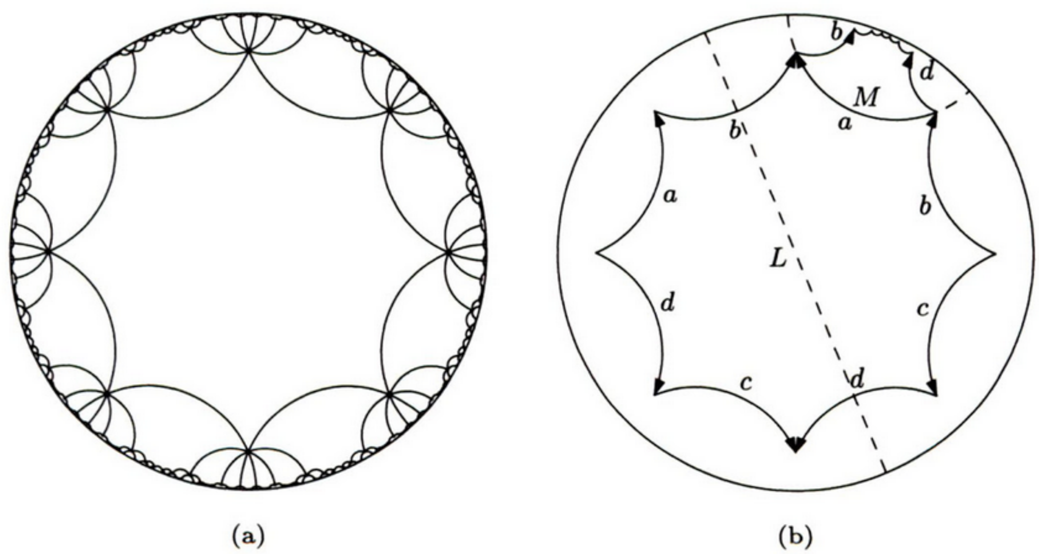


Figure 1.13. A tiling of the hyperbolic plane by regular octagons. (a) A tiling of the hyperbolic plane by identical regular octagons, seen in the Poincaré disk projection. **(b)** To get the small octagon from the big one, reflect in L, then in M.

The tiling of the hyperbolic plane by regular octagons used in Exercise 4.4(b).
Image credit: *Three-Dimensional Geometry and Topology, Vol. 1*, William P. Thurston, page 16.

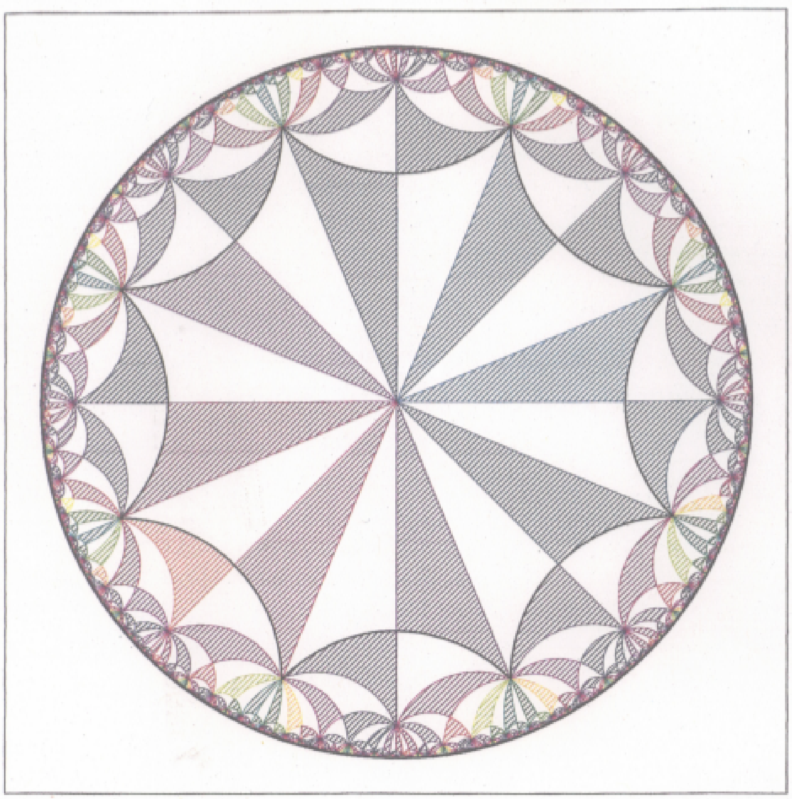
Exercise 4.5 (Conjugate covering space actions)

Let Y be a Hausdorff, locally path-connected, simply-connected space and let G and H be two discrete groups acting (on the right) on Y freely and properly discontinuously. Thus we have coverings $Y \to Y/G$ and $Y \to Y/H$ with deck transformation groups G and H respectively. Via their actions, we may consider G and H as subgroups of $\mathrm{Homeo}(Y)$, the group of all self-homeomorphisms of Y.
(1) If G and H are *conjugate* subgroups, then there is a homeomorphism $Y/G \cong Y/H$. (Suppose that $H = \phi G \phi^{-1}$. Then show that a homeomorphism may be defined by $y.G \mapsto y.\phi^{-1}.H$.)
(2) Conversely, show that if there is a homeomorphism $Y/G \cong Y/H$, that commutes with the two covering maps $Y \to Y/G$ and $Y \to Y/H$, then G and H must be conjugate. (Also give an example to show that the commutation condition is necessary.)

Exercise 4.6* (Coverings of $SO(3) \times SO(3)$)

(1) We know from Exercise 2.1 that $\mathbb{S}^n$ is simply-connected for $n \geqslant 2$. The action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{S}^n$ via the antipodal map is free and (vacuously, since it is a finite group) properly discontinuous. Thus $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$.
(2) Recall that there is a homeomorphism $\mathbb{RP}^3 \cong SO(3)$ (using Euler angles, for example). Thus $\pi_1(SO(3)) \cong \mathbb{Z}/2\mathbb{Z}$.
(3) Note that $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ has exactly five subgroups. Therefore, $SO(3) \times SO(3)$ has exactly five coverings: four of these are simply products of $\mathbb{S}^3$ and/or $SO(3)$.
(4) The remaining covering of $SO(3) \times SO(3)$ is the 2-fold covering corresponding to the subgroup $\{(0,0),(1,1)\}$ of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The general theory tells us that it is the quotient of the universal cover $\mathbb{S}^3 \times \mathbb{S}^3$ by the involution corresponding to the element $(1,1)$. This involution is given by $(x,y) \mapsto (-x,-y)$.
(5) Consider $\mathbb{S}^3$ as the unit sphere in the quaternions $\mathbb{H}$. Each element $(x,y) \in \mathbb{S}^3 \times \mathbb{S}^3$ therefore gives us a map $f_{(x,y)} \colon \mathbb{H} \to \mathbb{H}$ defined by $f_{(x,y)}(z) = xzy^{-1}$. Show that $f_{(x,y)}$ is a rotation, i.e., an element of $SO(4)$.
(6) Show also that every rotation in $SO(4)$ is of this form for some $(x,y) \in \mathbb{S}^3 \times \mathbb{S}^3$ and that two elements (x,y) and (x',y') induce the same rotation if and only if $(x',y') = (-x,-y)$. Deduce that the fifth covering space of $SO(3) \times SO(3)$ is homeomorphic to $SO(4)$.

Figure 12.3. A tiling of the hyperbolic plane modelled by the circular disk. From the hyperbolic point of view, all the 'triangles' are exactly the same shape. The (orientation-preserving) symmetries are Möbius maps which preserve this tiling and map any coloured triangle onto any other. This and the following figure were drawn by our DFS tiling program with the aid of an automaton provided by the program KBMAG by Derek Holt. The pattern of the colouring relates to the automaton.



A subdivision of the tiling of $\mathbb{H}^2$ by regular octagons, in which each octagon has been subdivided into 16 triangles.
Image credit: *Indra's Pearls*, David Mumford, Caroline Series and David Wright.