Aufgaben zur Topologie

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Wintersemester 2016/17

Week 2 — Winding number and vector fields on surfaces to be done by: 02.11.2016

Exercise 1.1 ("Two-dimensional intermediate value theorem")
Let $f\colon \mathbb{D}^2 \to \mathbb{R}^2$ be a smooth function and $z \in \mathbb{R}^2$ a point such that $z \notin f(S^1) = f(\partial\mathbb{D}^2)$. Recall the definition of the *winding number* w of the curve $f|_{S^1}\colon S^1 \to \mathbb{R}^2$ about the point z. Assuming that $w \neq 0$, show that there exists a point $x \in \mathbb{D}^2$ such that $f(x) = z$.

Exercise 1.2 (Linking number of a two-component link in $\mathbb{R}^3$)
Suppose $f, g\colon S^1 \to \mathbb{R}^3$ are two curves whose images $f(S^1)$ and $g(S^1)$ are disjoint. We consider $\mathbb{R}^3$ as a subspace of S^3 via one-point compactification, so f and g become curves in S^3.
(a) Show that there is a homotopy $f_t\colon S^1 \to S^3$ such that $f_0 = f$ and the image of f_1 is the z-axis together with the point at infinity.
(b) Show that there is a homotopy $g_t\colon S^1 \to S^3$ such that, for each $t \in [0,1]$, the two images $f_t(S^1)$ and $g_t(S^1)$ are disjoint.
(c) The curve g_1 therefore lies in $\mathbb{R}^3$ minus the z-axis. Let $\mathrm{pr}_{\{1,2\}}\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the projection onto the first two coordinates. Then $\mathrm{pr}_{\{1,2\}} \circ g_1$ is a curve in $\mathbb{R}^2$ disjoint from $(0,0)$. Define the *linking number* of f and g to be the winding number of this curve about $(0,0)$. Explain why this definition is independent of the choices made during the first two steps, i.e., the two homotopies that you constructed.

Exercise 1.3 (Local degree of vector fields)
For each of the following sets of integers, give a vector field on S^2 with precisely k zeros having these integers as its local indices.
(a) $k = 2$ $\{1, 1\}$
(b) $k = 1$ $\{2\}$
(c) $k = 4$ $\{1, 1, 1, -1\}$
(d) $k = 2$ $\{2, 0\}$
(e) $k = 7$ $\{1, 1, 1, 1, 1, -1, -2\}$

Exercise 1.4 (Symmetric vector fields on spheres)
In this exercise we will take the point of view that a tangent vector field on $S^2 \subset \mathbb{R}^3$ is a continuous function $v\colon S^2 \to \mathbb{R}^3$ with the property that x and $v(x)$ are orthogonal vectors for each $x \in S^2$.
(a) Suppose that $v(x) = v(-x)$ for each $x \in S^2$. If x is a zero of v, what is the relationship between the local degree of v at x and the local degree of v at $-x$?
(b) Now fix an angle $\theta \in [0, 2\pi]$ and suppose that for each $x \in S^2$ we have

$$v(-x) \;=\; \cos(\theta).v(x) + \sin(\theta).(x \times v(x)),$$

where $\times$ denotes the cross product in $\mathbb{R}^3$. If x is a zero of v, what is the relationship between the local degree of v at x and the local degree of v at $-x$? How does this relationship depend on θ?
(c) To write: show that if $v(-x)$ and $v(x)$ are related by complex conjugation (can this be formulated in terms of the cross product?) for all x, then no such vector field can exist on S^2. However, it is possible on the torus (for a symmetric embedding of the torus into $\mathbb{R}^3$) – give an example.

Exercise 1.5 ("Satz vom Igel" for higher-genus surfaces)
To be written...