Aufgaben zur Topologie I

Prof. Dr. C.-F. Bödigheimer

Wintersemester 2019/20

Blatt 14 no due date !

J.H.C. Whitehead (1904 - 1960), known for many contributions in algebraic topology, in particular in homotopy theory. His CW complexes are a very suitable category of spaces, both for homology theory and for homotopy theory.

Exercise 14.1 (Curve complexes)
Let $F = F_{g,r}$ be an orientable, compact, connected surface of genus g and r boundary curves. We consider homotopy classes of closed curves c on F; we suppose the curves are without self-intersections, and they are not null-homotopic, they are not homotopic to a bounary curve, and that $F - c$ is connected. By $[c]$ we denote their homotopy classes. We denote the set of these homotopy classes by $\mathcal{C}(F)$. A *cut system* is a set $S = \{[c_0], \ldots, [c_{n+1}]\}$ of homotopy classes of such curves, which have representatives c_i such that they are (1) disjoint, (2) not homotopic, and (3) the complement $F - \bigcup_{i=0}^{n} c_i$ is connected.

- Show that the set cut systems form an abstract simplicial scheme $\mathcal{S}_\bullet(F)$: obviously, $\mathcal{C}(F)$ is the set of vertices and the cut systems with $n+1$ elements are the n-simplices.
- Show that the group $\Gamma_{g,n} = \mathrm{Aut}(F)$ of homotopy classes of homotopy equivalences of F acts simplicially on $\mathcal{S}_\bullet(F)$.
- What is the isotropy subgroup of a vertex $[c]$?

Exercise 14.2 (CW complexes and coverings)
Let $\pi\colon \tilde{X} \to X$ be a covering. If X is a CW complex, then so is $\tilde{X}$.

Exercise 14.3 (Subcomplexes and quotient complexes of CW complexes)
A subcomplex Y of a CW complex X is a subspace, which is the union of closures $\bar{X}_\sigma$ of cells X_σ of X, such that theyform a CW complex. In other words: $Y = \bigcup_{\sigma \in \mathcal{S}'} X_\sigma$ is determined by a subset $\mathcal{S}' \subset \mathcal{S}$ of the cells of X with the property: If $\sigma \in \mathcal{S}'$ and $\bar{X}_\sigma \cap X_\tau \neq \emptyset$, then $\tau \in \mathcal{S}'$.

a) Show that Y is a CW complex in its own right.

b) Show that $Z = X/Y$ is a CW complex.

Exercise 14.4 (Weak topology on CW complexes)
In the definition of a CW complex X we characterized closed subsets $B \subset X$. We can equally well characterize open subset of X. Show the equivalence:

$B \subset X$ is closed (resp. open) in X $\iff$ For each $\sigma \in \mathcal{S}$ the intersection $B \cap \bar{X}_\sigma$ is closed (resp. open) in $\bar{X}_\sigma$.

Exercise 14.5 (Spaces with prescribed homology)

(a) For any natural number k and $n \geq 1$ let $f\colon \mathbb{S}^n \to \mathbb{S}^n$ of degree k. The space $M = M(\mathbb{Z}/k, n) := \mathbb{S}^n \cup_f \mathbb{D}^{n+1}$ is called a *Moore space*. Show that the reduced homology of M is $\mathbb{Z}/k$ in degree n and zero in all other degrees.

(b) Let $\mathbb{A}$ be a finitely generated abelian group. Construct a connected space $M = M(\mathbb{A}, n)$ such that its reduced homology is $\mathbb{A}$ in degree n and zero in all other degrees. (Use (a) and the Elementary Divisor Theorem.)

(c) Let $\mathbb{A}$ be an arbitrary abelian group. Construct a connected space $M = M(\mathbb{A}, n)$ such that its reduced homology is $\mathbb{A}$ in degree n and zero in all other degrees. (Use (b) and a limit argument.)

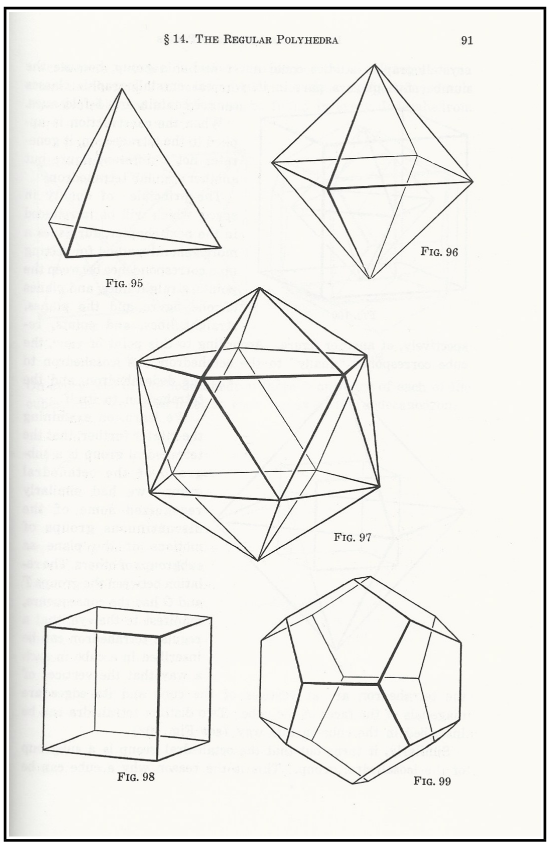
(d) Let $\mathbb{A}_1, \mathbb{A}_2, \ldots$ be a sequence of abelian groups. Construct a connected space M such that $H_n(M) \cong \mathbb{A}_n$ for all $n \geq 1$.

Galileo Galileis Grabmal in der Kirche Santa Croce in Florenz.

Exercise 14.6* (Platonic solids)
Already in ancient time it was known that there are only five Platonic solids. By this we mean a convex and compact subset of $\mathbb{R}^3$, whose boundary consists of regular n-polygons such that the intersection of any two is either empty or an edge, and at any vertex m edges meet.
Assume that the Euler characteristic (of the boundary) can be calculated as $E - K + F = 2$, where E, K, F denote the number of vertices, of edges and of polygons.

Show: *There are only five Platonic solids, namely tetrahedron, octahedron, hexahedron (or cube), dodecahedron and icosahedron.*

Hint: We have three equation: $E - K + F = 2$, and $mE = 2K$, and $nF = 2K$. And there are just five pairs (m, n) of solutions.



The five platonic solids, from D. Hilbert & S. Cohn-Vossen: *Geometry and the Imagination*, p. 92.