Aufgaben zur Topologie I

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Blatt 13 due by: 22. January 2020

K. Borsuk and P. Alexandroff, Radachówka, 1962

A. Granas, J. Dugundji: *Fixed Point Theory*, p. 105.

Exercise 13.1 (Additivity of the Euler characteristic)
Let $\mathbb{K}$ be a principal ideal domain. Using the Elementary Divisor Theorem we know for M any finitely generated $\mathbb{K}$-module that $M/\mathrm{tors}(M)$ is free and of finite rank, i.e. isomorphic to $\mathbb{K}^r$ for some well-defined number $r \in \mathbb{N}$; We call this number $r = \mathrm{rank}_{\mathbb{K}}(M)$ the *rank* of M (over $\mathbb{K}$).
Prove for A and B two finitely generated $\mathbb{K}$-modules:

- $\mathrm{tors}(A \oplus B) = \mathrm{tors}(A) \oplus \mathrm{tors}(B)$.
- $(A \oplus B)/\mathrm{tors}(A \oplus B) \cong A/\mathrm{tors}(A) \oplus B/\mathrm{tors}(B)$.
- $\mathrm{rank}_{\mathbb{K}}(A \oplus B) = \mathrm{rank}_{\mathbb{K}}(A) + \mathrm{rank}_{\mathbb{K}}(B)$.

Conclude for two graded $\mathbb{K}$-modules $A_\bullet$ and $B_\bullet$ of finite type the following formulas for the Euler characteristic and the Poincaré polynomial (wher we suppress $\mathbb{K}$ in the notation):

- $\chi(A_\bullet \oplus B_\bullet) = \chi(A_\bullet) + \chi(B_\bullet)$.
- $\mathfrak{P}_t(A_\bullet \oplus B_\bullet) = \mathfrak{P}_t(A_\bullet) + \mathfrak{P}_t(B_\bullet)$.

Exercise 13.2 (Mayer, Vietoris and Euler, or the additivity of the Euler characteristic for spaces.)
Let a space X be the union $X = X_1 \cup X_2$ of two open subspaces and assume X_1, X_2 and $X_0 = X_1 \cap X_2$ are of finite type over the principal ideal domain $\mathbb{K}$. We want to prove the following formula:

$$\chi(X_1 \cup X_2) \quad = \quad \chi(X_1) + \chi(X_2) - \chi(X_1 \cap X_2).$$

Clearly, we will use the long exact Mayer-Vietoris sequence.

(1) First, prove the following trick:
Let $E_\bullet : \quad \ldots \to E_{i-1} \to E_i \to E_{i+1} \to \ldots$ be a long exact sequence of finite type (that means: all modules are finitely generated and almost all are trivial), we can regard it as a chain complex, whose homology is trivial in each degree. Thus $\chi(E_\bullet) = 0$. Therefore, if we set $A_i := E_{3i}$, $B_i := E_{3i+1}$ and $C_i := E_{3i+2}$ for all $i \in \mathbb{Z}$, and consider the graded modules $A_\bullet$, $B_\bullet$ and $C_\bullet$ separately, we obtain

$$\chi(A_\bullet) - \chi(B_\bullet) + \chi(C_\bullet) = 0.$$

(2) Now apply this to the Mayer-Vietoris sequence.

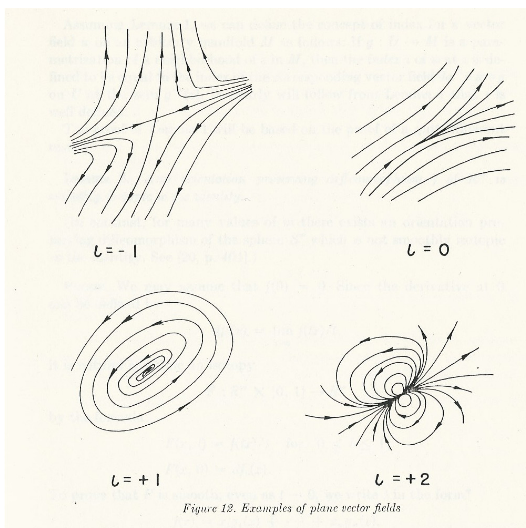


Figure 12. Examples of plane vector fields

J. Milnor: *Topology from a Differentiable Viewpoint*, p. 33.
The figures show vector fields in the plane and the indices of isolated zeroes.

Exercise 13.3 (Ham Sandwich Theorem)
Let $B_1, \ldots, B_n$ be bounded and Lebesgue measurable subsets of $\mathbb{R}^n$. There exists an affine (n-1)-plane A in $\mathbb{R}^n$, which cuts each B_i into two pieces of equal measure.

Prove this statement in five steps.

(1) Regard $\mathbb{R}^n$ as $\mathbb{R}^n \times \{1\} \subset \mathbb{R}^{n+1}$. Every affine (n-1)-plane A in $\mathbb{R}^n \times \{1\}$ determines a linear n-plane L in $\mathbb{R}^{n+1}$ by $L = \mathrm{Span}(A)$.

(2) Every linear n-plane L in $\mathbb{R}^{n+1}$ determines an affine (n-1)-plane A in $\mathbb{R}^n \times \{1\}$ by $A := L \cap \mathbb{R}^n \times \{1\}$.

(3) A unit vector $\zeta \in \mathbb{R}^{n+1}$ determines a linear n-plane in $\mathbb{R}^{n+1}$ by $L(\zeta) := \mathrm{Span}(\zeta)^\perp = \{x \in \mathbb{R}^{n+1} \mid \langle x, \zeta \rangle = 0\}$. Denote by $H(\zeta)$ the positive half-space of ζ, namely $H(\zeta) = \{x \in \mathbb{R}^{n+1} \mid \langle x, \zeta \rangle \geq 0\}$.

(4) Use (without proof) from analysis: For a bounded set $B \subset \mathbb{R}^n \times \{1\}$ the function $\zeta \mapsto \mu(H(\zeta) \cap B)$ is continuous. (Here μ is the Lebesgue measure in $\mathbb{R}^n \times \{1\}$.)

(5) Consider the map $f\colon \mathbb{S}^n \to \mathbb{R}^n$ defined by $f(\zeta) := (f_1(\zeta), \ldots, f_n(\zeta))$ with $f_i(\zeta) := \mu(H(\zeta) \cap B_i)$ and apply the Borsuk-Ulam Theorem.

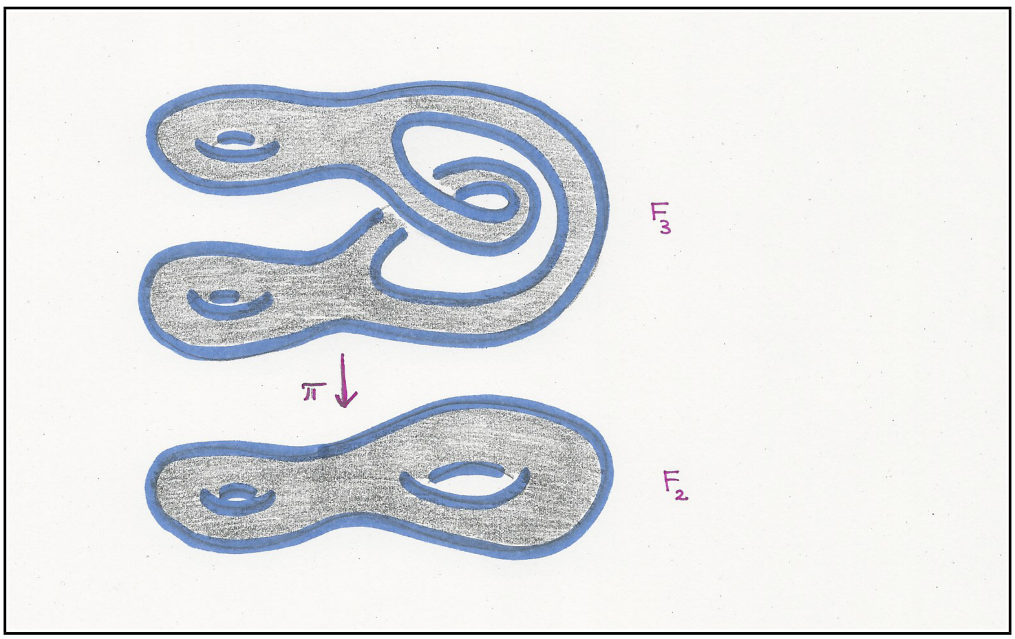
Exercise 13.4 (Equivalent versions of the Borsuk-Ulam Theorem)
Consider the two statements:

(I) *If* $f\colon \mathbb{S}^m \to \mathbb{S}^n$ *satisfies* $f(-x) = -f(x)$*, then* $m \leq n$*.*

(II) *For any* $f\colon \mathbb{S}^n \to \mathbb{R}^n$ *there is an* $x \in \mathbb{S}^n$ *such that* $f(x) = f(-x)$*.*

In the lecture we derived (II) from (I). Show that one can vice versa derive (I) from (II).

Exercise 13.5 (Maps between real projective spaces)
Let $f\colon \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^m$ any based map. If $0 < m < n$, the $f_*\colon \pi_1(\mathbb{R}\mathrm{P}^n) \to \pi_1(\mathbb{R}\mathrm{P}^m)$ is trivial.
Conclude: $\mathbb{R}\mathrm{P}^m \subset \mathbb{R}\mathrm{P}^n$ is not a retract for $0 < m < n$.



A 2-fold covering of a surface of genus 2 by a surface of genus 3.

Exercise 13.6* (Exact transfer sequence)

Consider the 2-fold covering $\pi\colon F_3 \to F_2$ shown in the figure above; F_g denotes a connected, orientable and closed surface of genus g. Study the exact transfer sequence of π in homology with coefficients in $\mathbb{Z}_2$:

$$0 \to H_2(F_2) \xrightarrow{\mathrm{Tr}(\pi)} H_2(F_3) \xrightarrow{\pi_*} H_2(F_2) \xrightarrow{\partial_*^T} H_1(F_2) \xrightarrow{\mathrm{Tr}(\pi)} H_1(F_3) \xrightarrow{\pi_*} H_1(F_2) \xrightarrow{\partial_*^T} H_0(F_2) \xrightarrow{\mathrm{Tr}(\pi)} H_0(F_3) \xrightarrow{\pi_*} H_0(F_2) \to 0\,.$$

Compute all homology groups, give generators, and compute all homomorphisms in the sequence.