Exercise 13.1 (Additivity of the Euler characteristic)

Let \( \mathbb{K} \) be a principal ideal domain. Using the Elementary Divisor Theorem we know for \( M \) any finitely generated \( \mathbb{K} \)-module that \( M/\text{tors}(M) \) is free and of finite rank, i.e. isomorphic to \( \mathbb{K}^r \) for some well-defined number \( r \in \mathbb{N} \); We call this number \( r = \text{rank}_\mathbb{K}(M) \) the rank of \( M \) (over \( \mathbb{K} \)).

Prove for \( A \) and \( B \) two finitely generated \( \mathbb{K} \)-modules:

1. \( \text{tors}(A \oplus B) = \text{tors}(A) \oplus \text{tors}(B) \).
2. \( (A \oplus B)/\text{tors}(A \oplus B) \cong A/\text{tors}(A) \oplus B/\text{tors}(B) \).
3. \( \text{rank}_\mathbb{K}(A \oplus B) = \text{rank}_\mathbb{K}(A) + \text{rank}_\mathbb{K}(B) \).

Conclude for two graded \( \mathbb{K} \)-modules \( A_* \) and \( B_* \) of finite type the following formulas for the Euler characteristic and the Poincaré polynomial (where we suppress \( \mathbb{K} \) in the notation):

1. \( \chi(A_* \oplus B_*) = \chi(A_*) + \chi(B_*) \).
2. \( \mathbb{P}(A_* \oplus B_*) = \mathbb{P}(A_*) + \mathbb{P}(B_*) \).
Exercise 13.2 (Mayer, Vietoris and Euler, or the additivity of the Euler characteristic for spaces.)
Let a space $X$ be the union $X = X_1 \cup X_2$ of two open subspaces and assume $X_1$, $X_2$ and $X_0 = X_1 \cap X_2$ are of finite type over the principal ideal domain $\mathbb{K}$. We want to prove the following formula:

$$\chi(X_1 \cup X_2) = \chi(X_1) + \chi(X_2) - \chi(X_1 \cap X_2).$$

Clearly, we will use the long exact Mayer-Vietoris sequence.

(1) First, prove the following trick:
Let $E_\bullet: \ldots \rightarrow E_{i-1} \rightarrow E_i \rightarrow E_{i+1} \rightarrow \ldots$ be a long exact sequence of finite type (that means: all modules are finitely generated and almost all are trivial), we can regard it as a chain complex, whose homology is trivial in each degree. Thus $\chi(E_\bullet) = 0$. Therefore, if we set $A_i := E_{3i}$, $B_i := E_{3i+1}$ and $C_i := E_{3i+2}$ for all $i \in \mathbb{Z}$, and consider the graded modules $A_\bullet$, $B_\bullet$ and $C_\bullet$ separately, we obtain

$$\chi(A_\bullet) - \chi(B_\bullet) + \chi(C_\bullet) = 0.$$

(2) Now apply this to the Mayer-Vietoris sequence.

Exercise 13.3 (Ham Sandwich Theorem)
Let $B_1, \ldots, B_n$ be bounded and Lebesgue measurable subsets of $\mathbb{R}^n$. There exists an affine (n-1)-plane $A$ in $\mathbb{R}^n$, which cuts each $B_i$ into two pieces of equal measure.

Prove this statement in five steps.

(1) Regard $\mathbb{R}^n$ as $\mathbb{R}^n \times \{1\} \subset \mathbb{R}^{n+1}$. Every affine (n-1)-plane $A$ in $\mathbb{R}^n \times \{1\}$ determines a linear n-plane $L$ in $\mathbb{R}^{n+1}$ by $L = \text{Span}(A)$.

(2) Every linear n-plane $L$ in $\mathbb{R}^{n+1}$ determines an affine (n-1)-plane $A$ in $\mathbb{R}^n \times \{1\}$ by $A := L \cap \mathbb{R}^n \times \{1\}$.

(3) A unit vector $\zeta \in \mathbb{R}^{n+1}$ determines a linear n-plane in $\mathbb{R}^{n+1}$ by $L(\zeta) := \text{Span}(\zeta) ^{\bot} = \{ x \in \mathbb{R}^{n+1} | \langle x, \zeta \rangle = 0 \}$. Denote by $H(\zeta)$ the positive half-space of $\zeta$, namely $H(\zeta) = \{ x \in \mathbb{R}^{n+1} | \langle x, \zeta \rangle \geq 0 \}$.

(4) Use (without proof) from analysis: For a bounded set $B \subset \mathbb{R}^n \times \{1\}$ the function $\zeta \mapsto \mu(H(\zeta) \cap B)$ is continuous. (Here $\mu$ is the Lebesgue measure in $\mathbb{R}^n \times \{1\}$.)

The figures show vector fields in the plane and the indices of isolated zeroes.
Consider the map $f: \mathbb{S}^n \to \mathbb{R}^n$ defined by $f(\zeta) := (f_1(\zeta), \ldots, f_n(\zeta))$ with $f_i(\zeta) := \mu(H(\zeta) \cap B_i)$ and apply the Borsuk-Ulam Theorem.

**Exercise 13.4** (Equivalent versions of the Borsuk-Ulam Theorem)
Consider the two statements:

(I) If $f: \mathbb{S}^m \to \mathbb{S}^n$ satisfies $f(-x) = -f(x)$, then $m \leq n$.

(II) For any $f: \mathbb{S}^n \to \mathbb{R}^n$ there is an $x \in \mathbb{S}^n$ such that $f(x) = f(-x)$.

In the lecture we derived (II) from (I). Show that one can vice versa derive (I) from (II).

**Exercise 13.5** (Maps between real projective spaces)
Let $f: \mathbb{R}P^n \to \mathbb{R}P^m$ any based map. If $0 < m < n$, the $f_*: \pi_1(\mathbb{R}P^n) \to \pi_1(\mathbb{R}P^m)$ is trivial.
Conclude: $\mathbb{R}P^m \subset \mathbb{R}P^n$ is not a retract for $0 < m < n$.

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A 2-fold covering of a surface of genus 2 by a surface of genus 3.

**Exercise 13.6** (Exact transfer sequence)

Consider the 2-fold covering $\pi: F_3 \to F_2$ shown in the figure above; $F_g$ denotes a connected, orientable and closed surface of genus $g$. Study the exact transfer sequence of $\pi$ in homology with coefficients in $\mathbb{Z}_2$:

$$0 \to H_2(F_2) \xrightarrow{\text{Tr}(\pi)} H_2(F_3) \xrightarrow{\pi_*} H_2(F_2) \xrightarrow{\partial_2^T} H_1(F_2) \xrightarrow{\text{Tr}(\pi)} H_1(F_3) \xrightarrow{\pi_*} H_1(F_2) \xrightarrow{\partial_1^T} H_0(F_2) \xrightarrow{\text{Tr}(\pi)} H_0(F_3) \xrightarrow{\pi_*} H_0(F_2) \to 0.$$ 

Compute all homology groups, give generators, and compute all homomorphisms in the sequence.