Aufgaben zur Topologie I

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Blatt 12 due by: 15. Januar 2020

Portrait of Leonhard Euler (1707 - 1783), by E. Handmann (1753).

Exercise 12.1 (Maps $\mathbb{S}^n \to \mathbb{S}^n$ without fixed points.)
Let $f\colon \mathbb{S}^n \to \mathbb{S}^n$ a map without fixed points. Show that f is homotopic to the antipodal map $A(z) = -z$.

Exercise 12.2 (The winding number I)
For a map $f\colon \mathbb{S}^n \to \mathbb{R}^{n+1}$ and a point $P \notin f(\mathbb{S}^n)$ we consider the map

$$F\colon \mathbb{S}^n \to \mathbb{S}^n, \quad F(x) := \frac{f(x) - P}{||f(x) - P||}$$

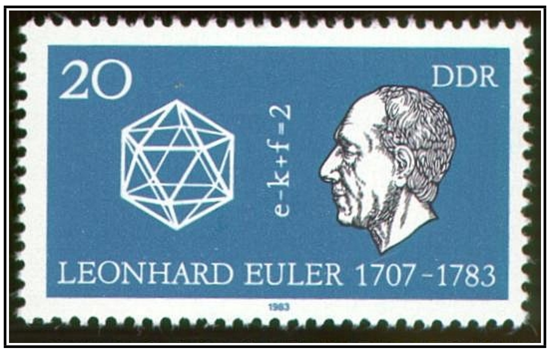
and define the *winding number of f around P* by $\mathrm{Um}(f, P) := \mathrm{grad}(F)$. Now show the following:

1. If $f \simeq g$ in $\mathbb{R}^{n+1} - P$, then $\mathrm{Um}(f, P) = \mathrm{Um}(g, P)$.
2. If f is in $\mathbb{R}^{n+1} - P$ homotopic to a constant map, then $\mathrm{Um}(f, P) = 0$.
3. If w is a path in the complement $\mathbb{R}^{n+1} - f(\mathbb{S}^n)$, then $\mathrm{Um}(f, w(0)) = \mathrm{Um}(f, w(1))$.

Exercise 12.3 (Winding number II)
Let $g\colon \mathbb{D}^{n+1} \to \mathbb{R}^{n+1}$ and denote its restriction to the boundary by f; assume P be in the complement of $f(\mathbb{S}^n)$.
<u>Kronecker knew</u>: If $\mathrm{Um}(f,P) \neq 0$, then there is a point $\zeta \in \mathbb{D}^{n+1}$ with $g(\zeta) = P$.

Exercise 12.4 (Euler Characteristic)
Let X be a connected space with homology $H_*(X;\mathbb{Q})$ of finite type.
a) Show that for each n the product $Y = \mathbb{S}^n \times X$ has also homology of finite type.
b) Show that

$$\chi(\mathbb{S}^n \times X) = 0 \quad \text{for } n \text{ odd} \tag{1}$$
$$\chi(\mathbb{S}^n \times X) = 2\,\chi(X) \quad \text{for } n \text{ even} \tag{2}$$



Stamp of 1983, to commemorate 200 years since Euler's death. It shows formula $e - k + f = 2$ for the Euler characteristic of polyhedra homeomorphic to the 2-sphere, here an icosahedron.

Exercise 12.5* (Poincaré polynomial I)
Let $M_\bullet$ be a graded module of finite type over a field $\mathbb{F}$. The *Poincaré polynomial* of $M_\bullet$ is defined as the Laurent polynomial $P_t(M_\bullet) := \sum_{k\in\mathbb{Z}} \dim(M_k)\,t^k$. Obviously, setting $t = -1$ gives the Euler characteristic: $P_{-1}(M_\bullet) = \chi(M_\bullet)$. Now let $C_\bullet$ be a chain complex of finite type over $\mathbb{F}$. The homology groups $H_k(C_\bullet)$ are then all of finite dimension. Show the following formula:

$$P_t(C_\bullet) - P_t(H_*(C_\bullet))) = (1+t)\,Q(t)\,,$$

where $Q(t)$ is a polynomial with coefficients in $\mathbb{N}$, namely $q_k = \dim(C_k) - \dim(H_k(C_\bullet))$.

Remark: If in $C_\bullet$ all modules C_k are finite-dimensional, but perhaps non-trivial in infinitely many degrees, then we get a Poincaré series instead of a Poincaré polynomial. Is the statement still true ?

Exercise 12.6* (Poincaré polynomial II)
Let (X,Y,Z) be a triple of spaces and assume that the pairs $X,Y)$, (Y,Z) an (X,Z) are all of finite type. (In fact, if any two of the pairs is of finite type, then so is the third.)
We set $P_t(X,Y) := P_t(H_*(X,Y))$, where we take homology with coefficients in a field $\mathbb{F}$.
Show that

$$P_t(X,Y) + P_t(Y,Z) = P_t(X,Z) + (1+t)Q(t)\,,$$

where $Q(t)$ is a polynomial with coefficients in $\mathbb{N}$.
(The corresponding remark as in Exercise 12.5 applies.)