Aufgaben zur Topologie I

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Blatt 11 due by: 08.01.2020

James Alexander (1888 - 1971), at a conference in Moscow 1935. He made important contributions in knot theory; the duality in Exc. 11.1 is an example of Alexander duality.

Exercise 11.1* (A kind of duality)
Let $G \subset \mathbb{S}^n$ be a finite, connected graph. Show that

$$\tilde{H}_k(\mathbb{S}^n - G) \cong \tilde{H}_{n-k-1}(G)$$

for all $k \geq 0$.

Hint: (a) Compute the homology of the (abstract) graph G: First do the case of a tree, which is contractible. Then consider the long exact homology sequence of the pair (G, T) and that you know the space G/T.
(b) Now for an embedded graph $G \subset \mathbb{S}^n$: First do the case of an embedded tree, by induction on the number of edges. Then start with a spanning tree T of G and proceed by induction on the number of edges outside the spanning tree.

Exercise 11.2* (Linking numbers I)
Let $A \subset \mathbb{S}^n$ be an enbedded sphere of dimension $p < n$. We know: $H_q(\mathbb{S}^n - A) \cong \mathbb{Z}$ for $q = n - p - 1$; so we can choose some generator w of this group. Let $f\colon \mathbb{S}^q \to \mathbb{S}^n$ be an embedding of a sphere of dimension $q = n - p - 1$ with image $B = f(\mathbb{S}^q)$ disjoint from A. In this situation we can define an integer L by the equation

$$f_*(\omega_q) = L\, w,$$

where $\omega_q \in H_q(\mathbb{S}^q)$ is a generator. We call this number $L = \mathrm{Link}(A, B)$ the *linking number* of A and B. Remark: The name obviously comes from the situation $n = 3$ and $p = q = 1$, where we measure the linking of two closed and disjoint curves in 3-space. Show the following:

a) $\mathrm{Link}(A, B)$ depends on the choice of the two generators, but only up to sign.

b) $\mathrm{Link}(A, B) = \pm\, \mathrm{Link}(B, A)$, despite the asymmetric definition.

c) $\mathrm{Link}(A, B) = 0$, if A and B lie on different sides of a hyperplane.

d) $\mathrm{Link}(A, B) = \pm 1$, if A is given as the intersection of $\mathbb{S}^n$ with the subset $x_1 = \ldots = x_{p+1} = 0$, and B is given as the intersection of $\mathbb{S}^n$ with the subset $x_{n-q+1} = \ldots = x_{n+1} = 0$, both times in $\mathbb{R}^{n+1}$.

Exercise 11.3* (Linking numbers II)
Let $f\colon \mathbb{S}^p \to \mathbb{R}^n$ and $g\colon \mathbb{S}^q \to \mathbb{R}^n$ be two disjoint embeddings, where $p + q = n - 1$. We can consider the difference $f(x) - g(y)$ and then normalize it, obtaining a map

$$\Delta(f, g)\colon \mathbb{S}^p \times \mathbb{S}^q \longrightarrow \mathbb{S}^{n-1}, \quad \Delta(f, g)(x, y) := \frac{f(x) - g(y)}{\|f(x) - g(y)\|}\,.$$

Since we know $H_{n-1}(\mathbb{S}^p \times \mathbb{S}^q) \cong \mathbb{Z}$, we can choose a generator $u_{p,q}$. Likewise, we choose a generator ω_{n-1} in $H_{n-1}(\mathbb{S}^{n-1})$. The equation

$$\Delta(f, g)_*(u_{p,q}) = D\, \omega_{n-1}$$

determines an integer $D = D(f, g)$, which we may call the *degree* of $\Delta(f, g)$.
(Again, it is helpful to look at the case $n = 3$, $p = q = 1$ first: Two astronomers, one on a planet and one on a comet, calculate the difference betwen their positions and want to find out, if their paths are linked or not.)

a) $D(f, g)$ depends on the choice of the generators, but only up to sign.

b) $D(f, g) = (-1)^n D(g, f)$.

c) $D(f, g) = 0$, if the images $A = f(\mathbb{S}^p)$ and $B = g(\mathbb{S}^q)$ lie on different sides of a hyperplane.

d) $D(f, g) = D(f', g')$, if $f \simeq f'$ or $g \simeq g'$ through homotopies f_t resp. g_t with disjoint images for each t.

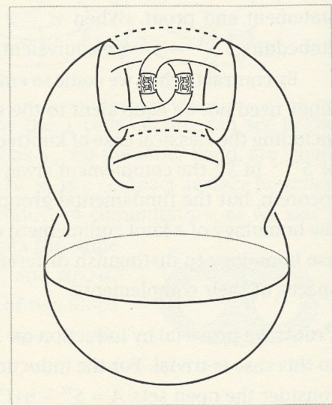
e) $D(f, g) = \pm\, \mathrm{Link}(A, B)$ for $A = f(\mathbb{S}^p)$ and $B = g(\mathbb{S}^q)$.

Exercise 11.4* (Division algebras)
Prove the following
Theorem: *If $n > 0$, then $\mathbb{R}^{2n+1}$ does not have the structure of a division algebra over $\mathbb{R}$, in particular not a field extension of $\mathbb{R}$.*

Hint: Assume it is a division algebra. For a non-zero element $a \in \mathbb{R}^{2n+1}$ consider the two maps

$$f_\pm\colon \mathbb{S}^{2n} \longrightarrow \mathbb{S}^{2n} \quad \text{defined by} \quad f_\pm(x) = \pm\, \frac{a \cdot x}{\|a \cdot x\|}$$

Example 2B.2: The Alexander Horned Sphere. This is a subspace $S \subset \mathbb{R}^3$ homeomorphic to S^2 such that the unbounded component of $\mathbb{R}^3 - S$ is not simply-connected. We will construct a sequence of compact subspaces $X_0 \supset X_1 \supset \cdots$ of $\mathbb{R}^3$ whose intersection X is homeomorphic to D^3, and S will be the boundary sphere of X. To begin, X_0 is a solid torus $S^1 \times D^2$ obtained from a ball B_0 by attaching a handle $I \times D^2$ along $\partial I \times D^2$. In the figure this handle is shown as the union of two 'horns' attached to the ball, together with a shorter handle drawn as dashed lines. To form the space $X_1 \subset X_0$ we delete part of the short handle, so that what remains is a pair of linked handles attached to the ball B_1 that is the union of B_0 with the two horns. To form X_2 the process is repeated: Decompose each of the second stage handles as a pair of horns and a short handle, then delete a part of the short handle. In the same way X_n is constructed inductively from X_{n-1}. Thus X_n is a ball B_n with 2^n handles attached, and B_n is obtained from B_{n-1} by attaching 2^n



This is a picture of Alexander's horned sphere, named after J. Alexander (1888 - 1971), an American topologist. This is the embedding of a 2-sphere into $\mathbb{R}^3$; the unbounded component is acyclic, but not simply-connected; so its fundamental group becomes trivial, when abelianized. — From A.Hatcher: *Algebraic Topology*, pp. 170/171, see continuation below.

and compare their degrees.

Exercise 11.5* (Embeddings of surfaces into surfaces)
Show that a closed, compact surface of genus g can not be embedded into one of genus g', unless $g = g'$.

horns. There are homeomorphisms $h_n : B_{n-1} \to B_n$ that are the identity outside a small neighborhood of $B_n - B_{n-1}$. As n goes to infinity, the composition $h_n \cdots h_1$ approaches a map $f : B_0 \to \mathbb{R}^3$ which is continuous since the convergence is uniform. The set of points in B_0 where f is not equal to $h_n \cdots h_1$ for large n is a Cantor set, whose image under f is the intersection of all the handles. It is not hard to see that f is one-to-one. By compactness it follows that f is a homeomorphism onto its image, a ball $B \subset \mathbb{R}^3$ whose boundary sphere $f(\partial B_0)$ is S, the Alexander horned sphere.

Exercise 11.6* (Möbius band and $\mathbb{R}^2$)
Can the Möbius band be embedded into $\mathbb{R}^2$? (— Don't argue with orientability; this notion and theorems about orientable and non-orientable manifolds are not yet available to us.)