Aufgaben zur Topologie I

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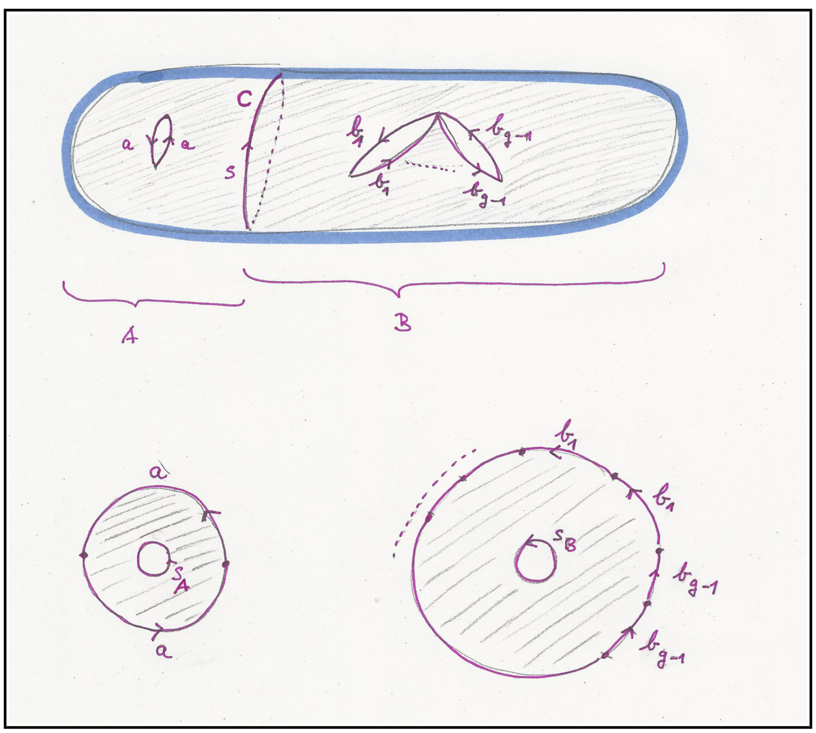
Blatt 9 due by: 11. 12. 2019

Exercise 9.1 (Suspension isomorphism for a homology theory)
Let h_* denote a homology theory. Prove from the Eilenberg-Steenrod axioms alone that there is for every space X and all $n \in \mathbb{Z}$ a suspension isomorphism

$$\sigma\colon h_n(X,*) \longrightarrow h_{n+1}(\Sigma X,*)$$

where ΣX is the unreduced suspension and $h_*(-,*)$ denotes the relative homology with respect to a point.
Assume that h_* satisfies the dimension axiom with coefficient group $h_0(pt) = G$. Conclude that $h_n(\mathbb{S}^m,*) \cong G$ if $n = m$, and $h_n(\mathbb{S}^m,*) = 0$ else.



A non-orientable surface of genus $g \geq 1$.

Exercise 9.2 (Homology of non-orientable surfaces)
A non-orientable surface $N = N_g$ of genus $g \geq 1$ is a connected sum of projective planes; see the figure above, the

notation of which we use throughout. We have the decomposition $N = A \cup B$ and $C = A \cap B$:

$$\begin{tikzcd} & N & \\ A \arrow[ur, "\kappa_A"] & & B \arrow[ul, "\kappa_B"'] \\ & C \arrow[ul, "\iota_A"] \arrow[ur, "\iota_B"'] & \end{tikzcd}$$

First, compute the homology of the three parts:

(1) $A \simeq \mathbb{S}^1$ and $[a] \in H_1(A) \cong \mathbb{Z}$ is a generator.

(2) $B \simeq \mathbb{S}^1 \vee \ldots \vee \mathbb{S}^1$ (with $g-1$ copies of $\mathbb{S}^1$) and $[b_1], \ldots, [b_{g-1}] \in H_1(B)$ are generators.

(3) $C \cong \mathbb{S}^1$ and $[s] \in H_1(C)$ is a generator.

Let us write $s_A := \iota_A(s)$ and $s_B := \iota_B(s)$. Next prove the following statements, using a triangulation of the spaces A and B.

(4) There is a 2-chain $\alpha \in S_2(A)$ with boundary $\partial(\alpha) = s_A + 2a$ and there is 2-chain $\beta \in S_2(B)$ with boundary $\partial(\beta) = s_B + 2b_1 + \ldots b_{g-1}$.

(5) Thus for the homology classes we have $\iota_{A*}([s]) = 2[a]$ in $H_1(A)$ and $\iota_{B*}([s]) = [s_B] = 2[b_1] + \ldots 2[b_{g-1}]$. Conclude that $J_1 = (\iota_{A*}, \iota_{B*})$ is injective.

Now we look at the Mayer-Vietoris sequence:

$$\begin{tikzcd}[column sep=small] \cdots & \cdots & \cdots \arrow[dll, "\partial_*"'] & \\ H_3(C) \arrow[r, "J_3"'] & H_3(A) \oplus H_3(B) \arrow[r, "K_3"] & H_3(N) \arrow[dll, "\partial_*"'] & \\ H_2(C) \arrow[r, "J_2"'] & H_2(A) \oplus H_2(B) \arrow[r, "K_2"] & H_2(N) \arrow[dll, "\partial_*"'] & \\ H_1(C) \arrow[r, "J_1"'] & H_1(A) \oplus H_1(B) \arrow[r, "K_1"] & H_1(N) \arrow[dll, "\partial_*"'] & \\ H_0(C) \arrow[r, "J_0"'] & H_0(A) \oplus H_0(B) \arrow[r, "K_0"] & H_0(N) \arrow[r] & 0 \end{tikzcd}$$

The diagonal arrows are the connecting homomorphism in the Mayer Vietoris sequence; and recall that $J_i = (\iota_{A*}, \iota_{B*})$ and $K_i = \kappa_{A*} - \kappa_{B*}$ in degree i. Strike out all terms which are trivial. Write down all non-trivial terms we already know. Then prove the following:

(6) $H_n(N_g) = 0$ for $n \geq 3$.

(7) J_0 is mono and thus the connecting homomorphism $\partial_* \colon H_1(N) \to H_0(C)$ is trivial. So the last three terms form a short exact sequence

$$0 \to H_0(C) \to H_0(A) \oplus H_0(B) \to H_0(N) \to 0.$$

of free groups of rank 1, 2 and 1, resp.. Clearly $H_0(N) = \mathbb{Z}$.

(8) K_1 is epi.

(9) $J_1([s]) = (2[a], 2[b_1] + \ldots + 2[b_{g-1}])$. Thus it is mono.

(10) Conclude $H_2(N) = 0$.

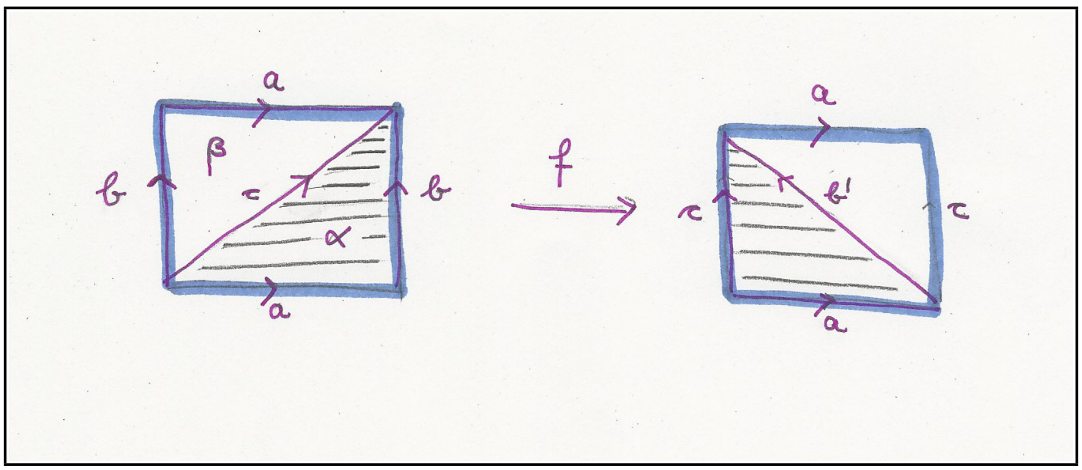
(11) There is a short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{J_1} \mathbb{Z} \oplus \mathbb{Z}^{g-1} \xrightarrow{K_1} H_1(N) \longrightarrow 0$$

from which you can compute $H_1(N)$, remembering the Elementary Divisor Theorem. Beware, — it is not $(\mathbb{Z}/2\mathbb{Z})^g$.

Exercise 9.3 (Self-maps of the torus)
Let $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$ denote the torus, the homology of which we know: Refering to the figure below:
(0) $H_0(\mathbb{T}) = \mathbb{Z}$;
(1) $H_1(\mathbb{T}) \cong \mathbb{Z} \oplus \mathbb{Z}$ with generators $[a]$, $[b]$ represented by the curves a, b; and
(2) $H_2(\mathbb{T}) \cong \mathbb{Z}$ with generator represented by the 2-chain $\alpha - \beta$ given by the two triangles.

Let $f\colon \mathbb{T} \to \mathbb{T}$ be the self-map as in the figure: it is determined by $f(a) = a$, $f(b) = b'$ and $f(c) = b$. Thus it fixes the bottom side a of the shaded triangle and slides its top point to the left, and it fixes the top side a of the unshaded triangle and slides its bottom point to the right. It seems prima vista not continuous along c, but it is continuous.
Compute $f_*\colon H_i(\mathbb{T}) \to H_i(\mathbb{T})$ in all degrees i.
Remark: Such a map is called a Dehn-twist. They generate the mapping class group of any surface, which is the group of isotopy classes of orientation-preserving homeomorphisms.



A Dehn-twist f on a torus.

Exercise 9.4 (Invariance of the boundary)
Let M and N denote manifolds of the same dimension; we allow both to have non-empty boundary ∂M resp. ∂N. Prove: Any homeomorphism $f\colon M \to N$ must send the boundary of M to the boundary of N; furthermore, f restricts to a homeomorphism $g = f|\colon \partial M \to \partial N$.

Exercise 9.5 (Brouwer Fix Point Theorem)
Use homology groups to prove: **Every continuous self-map $f\colon \mathbb{D}^n \to \mathbb{D}^n$ has at least one fixed point.**

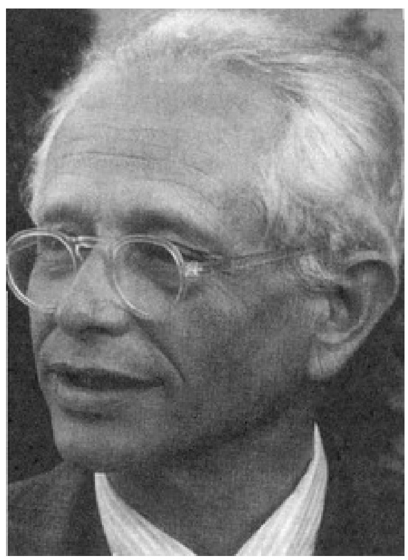
Exercise 9.6* (Mapping tori)
Let $\varphi\colon F \to F$ be any map. Its *mapping torus* is the space $T(\varphi) = (F \times [0,1])/\sim$, where $(x,0) \sim (\varphi(x),1)$. There

is an obvious map $\pi\colon T(\varphi) \to [0,1]/(0 \sim 1) = \mathbb{S}^1$. The inverse image of each point $t \in \mathbb{S}^1$ is homeomorphic to F. If the map φ is a homeomorphism, $T(\varphi)$ is a so-called fibre bundle over $\mathbb{S}^1$ with fiber F.
Decompose $\mathbb{S}^1 = \mathbb{S}^1_+ \cup \mathbb{S}^1_-$ into an upper and lower hemisphere and correspondingly $E = T(\varphi)$ into $E_+ = \pi^{-1}(\mathbb{S}^1_+)$ and $E_- = \pi^{-1}(\mathbb{S}^1_-)$.
(a) Use the Mayer-Vietoris sequence to show that there is a long exact sequence of the form

$$\ldots \to H_n(F) \xrightarrow{\mathrm{id} - \varphi_*} H_n(F) \to H_n(E) \to H_{n-1}(F) \to \ldots$$

Examples:
(1) If $\varphi = \mathrm{id}_F$, then $T(\varphi) \cong F \times \mathbb{S}^1$.
(2) If $F = [0,1]$ and $\varphi(t) = 1 - t$, then $T(\varphi)$ is a Möbius band.
(3) If F is discrete with n points and φ some permutation, then $T(\varphi)$ is an n-fold covering of $\mathbb{S}^1$ with as many components as φ has cycles (counting the fixed points as 1-cycles).
(4) If $F = \mathbb{S}^1$ and $\varphi(x,y) = (-x,y)$, then $T(\varphi)$ is the Klein bottle.

(b) Determine the homology of two of theses examples.



Max Dehn (1878 - 1952) made important contributions in low-dimensional topology.