Exercise 9.1 (Suspension isomorphism for a homology theory)
Let $h_\ast$ denote a homology theory. Prove from the Eilenberg-Steenrod axioms alone that there is for every space $X$ and all $n \in \mathbb{Z}$ a suspension isomorphism

$$\sigma : h_n(X, \ast) \rightarrow h_{n+1}(\Sigma X, \ast)$$

where $\Sigma X$ is the unreduced suspension and $h_\ast(-, \ast)$ denotes the relative homology with respect to a point. Assume that $h_\ast$ satisfies the dimension axiom with coefficient group $h_0(pt) = G$. Conclude that $h_n(S^m, \ast) \cong G$ if $n = m$, and $h_n(S^m, \ast) = 0$ else.

Exercise 9.2 (Homology of non-orientable surfaces)
A non-orientable surface $N = N_g$ of genus $g \geq 1$ is a connected sum of projective planes; see the figure above, the

A non-orientable surface of genus $g \geq 1$. 
notation of which we use throughout. We have the decomposition $N = A \cup B$ and $C = A \cap B$:

$$
\begin{array}{c}
N \\
\kappa_A \\
A \\
\uparrow \\
\kappa_B \\
B \\
\downarrow \\
\downarrow \\
C \\
i_A \\
\uparrow \\
i_B \\
\end{array}
$$

First, compute the homology of the three parts:

1. $A \simeq \mathbb{S}^1$ and $[a] \in H_1(A) \cong \mathbb{Z}$ is a generator.

2. $B \simeq \mathbb{S}^1 \vee \ldots \vee \mathbb{S}^1$ (with $g - 1$ copies of $\mathbb{S}^1$) and $[b_1], \ldots, [b_{g-1}] \in H_1(B)$ are generators.

3. $C \cong \mathbb{S}^1$ and $[s] \in H_1(C)$ is a generator.

Let us write $s_A := \iota_A(s)$ and $s_B := \iota_B(s)$. Next prove the following statements, using a triangulation of the spaces $A$ and $B$.

4. There is a 2-chain $\alpha \in S_2(A)$ with boundary $\partial(\alpha) = s_A + 2a$ and there is 2-chain $\beta \in S_2(B)$ with boundary $\partial(\beta) = s_B + 2b_1 + \ldots b_{g-1}$.

5. Thus for the homology classes we have $\iota_A([s]) = 2[a]$ in $H_1(A)$ and $\iota_B([s]) = [s_B] = 2[b_1] + \ldots 2[b_{g-1}]$. Conclude that $J_1 = (\iota_A, \iota_B)$ is injective.

Now we look at the Mayer-Vietoris sequence:

$$
\begin{array}{ccccccccc}
\cdots & \cdots & \cdots & \cdots & H_3(C) & \xrightarrow{J_3} & H_3(A) \oplus H_3(B) & \xrightarrow{K_3} & H_3(N) \\
& & & & \downarrow \partial_3 & & \downarrow \partial_3 & & \\
& & & & H_2(C) & \xrightarrow{J_2} & H_2(A) \oplus H_2(B) & \xrightarrow{K_2} & H_2(N) \\
& & & & \downarrow \partial_2 & & \downarrow \partial_2 & & \\
& & & & H_1(C) & \xrightarrow{J_1} & H_1(A) \oplus H_1(B) & \xrightarrow{K_1} & H_1(N) \\
& & & & \downarrow \partial_1 & & \downarrow \partial_1 & & \\
& & & & H_0(C) & \xrightarrow{J_0} & H_0(A) \oplus H_0(B) & \xrightarrow{K_0} & H_0(N) & \rightarrow 0
\end{array}
$$

The diagonal arrows are the connecting homomorphism in the Mayer Vietoris sequence; and recall that $J_i = (\iota_{A*}, \iota_{B*})$ and $K_i = \kappa_{A*} - \kappa_{B*}$ in degree $i$. Strike out all terms which are trivial. Write down all non-trivial terms we already know. Then prove the following:

6. $H_n(N_g) = 0$ for $n \geq 3$.

7. $J_0$ is mono and thus the connecting homomorphism $\partial_* : H_1(N) \rightarrow H_0(C)$ is trivial. So the last three terms form a short exact sequence

$$
0 \rightarrow H_0(C) \rightarrow H_0(A) \oplus H_0(B) \rightarrow H_0(N) \rightarrow 0.
$$

of free groups of rank 1, 2 and 1, resp.. Clearly $H_0(N) = \mathbb{Z}$.

8. $K_1$ is epi.
(9) \( J_1([s]) = (2[a], 2[b_1] + \ldots + 2[b_{g-1}]). \) Thus it is mono.

(10) Conclude \( H_2(N) = 0. \)

(11) There is a short exact sequence

\[
0 \rightarrow \mathbb{Z} \xrightarrow{J_1} \mathbb{Z} \oplus \mathbb{Z}^{g-1} \xrightarrow{K_1} H_1(N) \rightarrow 0
\]

from which you can compute \( H_1(N) \), remembering the Elementary Divisor Theorem. Beware, — it is not \((\mathbb{Z}/2\mathbb{Z})^g\).

**Exercise 9.3** (Self-maps of the torus)

Let \( \mathbb{T} = S^1 \times S^1 \) denote the torus, the homology of which we know: Refering to the figure below:

(0) \( H_0(\mathbb{T}) = \mathbb{Z}; \)

(1) \( H_1(\mathbb{T}) \cong \mathbb{Z} \oplus \mathbb{Z} \) with generators \([a], [b]\) represented by the curves \( a, b \); and

(2) \( H_2(\mathbb{T}) \cong \mathbb{Z} \) with generator represented by the 2-chain \( \alpha - \beta \) given by the two triangles.

Let \( f : \mathbb{T} \rightarrow \mathbb{T} \) be the self-map as in the figure: it is determined by \( f(a) = a, f(b) = b' \) and \( f(c) = b \). Thus it fixes the bottom side \( a \) of the shaded triangle and slides its top point to the left, and it fixes the top side \( a \) of the unshaded triangle and slides its bottom point to the right. It seems prima vista not continuous along \( c \), but it is continuous.

Compute \( f^* : H_i(\mathbb{T}) \rightarrow H_i(\mathbb{T}) \) in all degrees \( i \).

Remark: Such a map is called a Dehn-twist. They generate the mapping class group of any surface, which is the group of isotopy classes of orientation-preserving homeomorphisms.

A Dehn-twist \( f \) on a torus.

**Exercise 9.4** (Invariance of the boundary)

Let \( M \) and \( N \) denote manifolds of the same dimension; we allow both to have non-empty boundary \( \partial M \) resp. \( \partial N \). Prove: Any homeomorphism \( f : M \rightarrow N \) must send the boundary of \( M \) to the boundary of \( N \); furthermore, \( f \) restricts to a homeomorphism \( g = f| : \partial M \rightarrow \partial N \).

**Exercise 9.5** (Brouwer Fix Point Theorem)

Use homology groups to prove: **Every continuous self-map** \( f : \mathbb{D}^n \rightarrow \mathbb{D}^n \) **has at least one fixed point.**

**Exercise 9.6** (Mapping tori)

Let \( \varphi : F \rightarrow F \) be any map. Its **mapping torus** is the space \( T(\varphi) = (F \times [0,1])/\sim \), where \( (x, 0) \sim (\varphi(x), 1) \). There
is an obvious map \( \pi: T(\varphi) \to [0,1]/(0 \sim 1) = S^1 \). The inverse image of each point \( t \in S^1 \) is homeomorphic to \( F \).
If the map \( \varphi \) is a homeomorphism, \( T(\varphi) \) is a so-called fibre bundle over \( S^1 \) with fiber \( F \).
Decompose \( S^1 = S^1_+ \cup S^1_- \) into an upper and lower hemisphere and correspondingly \( E = T(\varphi) \) into \( E_+ = \pi^{-1}(S^1_+) \) and \( E_- = \pi^{-1}(S^1_-) \).
(a) Use the Mayer-Vietoris sequence to show that there is a long exact sequence of the form

\[
\cdots \to H_n(F) \xrightarrow{id - \varphi_*} H_n(F) \to H_n(E) \to H_{n-1}(F) \to \cdots
\]

Examples:
(1) If \( \varphi = id_F \), then \( T(\varphi) \cong F \times S^1 \).
(2) If \( F = [0,1] \) and \( \varphi(t) = 1 - t \), then \( T(\varphi) \) is a Möbius band.
(3) If \( F \) is discrete with \( n \) points and \( \varphi \) some permutation, then \( T(\varphi) \) is an \( n \)-fold covering of \( S^1 \) with as many components as \( \varphi \) has cycles (counting the fixed points as 1-cycles).
(4) If \( F = S^1 \) and \( \varphi(x,y) = (-x,y) \), then \( T(\varphi) \) is the Klein bottle.

(b) Determine the homology of two of these examples.

Max Dehn (1878 - 1952) made important contributions in low-dimensional topology.