Aufgaben zur Topologie I

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The foto from 1932 in Zürich shows Luitzen Egbertus Jan Brouwer (1881 - 1966) on the right and Harald Bohr (1887 - 1951) on the left. Brouwer proved not only the famous fixed point theorem named after him; he also defined the degree for selfmaps of spheres, showed the invariance of the dimension and invariance of domain, and he founded intuitionism in the philosophy of mathematics. Harald Bohr is the younger brother of the famous physicist Niels Bohr; he himself is known for contributions to complex functions and analytic number theory.

Exercise 8.1 (Barycentric subdivision)
Consider the affine inclusion $b\colon \Delta^n \to \Delta^n$ which sends a basis vector e_k for $k = 0, \dots, n$ to

$$b(e_k) = \frac{1}{k+1}\left(\sum_{i=0}^{k} e_i\right).$$

The symmetric group $\mathfrak{S}_n^+ := \mathrm{Sym}\{0, 1, \dots, n\}$ of the letters $0, 1, \dots, n$ acts on Δ^n by permuting the coordinates, this means $\pi_\bullet(t_0, \dots, t_n) = (t_{\pi^{-1}(0)}, \dots, t_{\pi^{-1}(n)})$ for $\pi \in \mathfrak{S}_n^+$.

If we set $b^\pi(t) := \pi_\bullet b(t)$, then $b^\pi(e_k) = \frac{1}{k+1}\sum_{i=0}^{k} e_{\pi(i)}$.

Recall the barycentric operator $B\colon S_n(\Delta^M) \to S_n(\Delta^M)$, recursively defined by

$$B(\alpha) = \alpha \qquad \text{for } |\alpha| = 0, \tag{1}$$

$$B(\alpha) = (-1)^n K_v(B(\partial(\alpha))) = \sum_{i=0}^{n} (-1)^{n+i} K_v(B(\partial_i(\alpha))) \qquad \text{for } |\alpha| \geq 1, \tag{2}$$

where $v = \mathrm{bar}(\alpha) = \alpha(\frac{1}{n+1}\sum e_i)$ is the barycenter of α and $K_v\colon S_n(\Delta^M) \to S_{n+1}(\Delta^M)$ denotes the cone operator with cone point $v \in \Delta^M$.
With our preparations above we define for each permutation π an operator

$$B^\pi\colon S_n(X) \to S_n(X), \quad B^\pi(\alpha) := \alpha \circ b^\pi.$$

Show that if $\alpha\colon \Delta^n \to \Delta^m$ is an affine map, then $B(\alpha) = \sum_{\pi \in \mathfrak{S}_n^+} \mathrm{sign}(\pi)\, B^\pi(\alpha)$.

Exercise 8.2 (New homology theories)
We want to show: **Given any homology theory $h_*(X, A)$ and any torsion-free abelian group G, then $h'_*(X, A) := h_*(X; A) \otimes_{\mathbb{Z}} G$ is again a homology theory.**
You need the following statements about abelian groups.

(1) Every abelian group is the filtered[1] colimit (diret limit) of its finitely generated subgroups.

(2) Every torsion-free abelian group is the filtered colimit of free abelian subgroups of finite rank.

(3) Tensor products commute with filtered colimits: $A \otimes_{\mathbb{Z}} (\mathrm{colim}\, G_i) \cong \mathrm{colim}(A \otimes_{\mathbb{Z}} G_i)$.

(4) The filtered colimit of exact sequences is an exact sequence.

(5) Tensoring with a torsion-free abelian group is an exact functor, i.e., it preserves exact sequences.

(1) is obvious, and (2) follows easily from (1); Instead of proving them, write $\mathbb{Q}$ as a colimit of countably many $\mathbb{Z}$. Prove (3) and (4). And (5) is an easy consequence of (3) and (4).

Conclude now that $h'_*(X, A) := h_*(X; A) \otimes_{\mathbb{Z}} G$ is a homology theory, if $h_*(X, A)$ is one.

Remark: $H_*(X, A; \mathbb{Q})$, singular homology with coefficients in $\mathbb{Q}$, is a homology theory. And by the above, we know $H_*(X, A; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is also a homology theory. Are they equal ? — We will see later, that there is a natural transformation $T\colon H_*(X, A; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow H_*(X, A; \mathbb{Q})$, which is an isomorphism for all pairs (X, A).

Exercise 8.3 (Degree of a map $f\colon \mathbb{S}^n \to \mathbb{S}^n$)
The degree of a self-map $f\colon \mathbb{S}^n \to \mathbb{S}^n$ of a sphere is defined by the equation

$$f_*(\omega_n) = \deg(f)\, \omega_n\,,$$

where $\omega_n \in H_n(\mathbb{S}^n) \cong \mathbb{Z}$ is a generator. Show the following properties:

(1) $\deg(\mathrm{id}) = 1$.

(2) $\deg(f) = 0$, if f is homotopic to a constant map. (Example: f not surjective.)

(3) $\deg(f \circ g) = \deg(f) \deg(g)$

(4) $\deg(f) = \pm 1$, if f is a homotopy-equivalence.

(5) $\deg(\Sigma'(f)) = \deg(f)$ for the unreduced suspension.

(6) For each $k \in \mathbb{Z}$ and $n \geq 1$ there is a map $f\colon \mathbb{S}^n \to \mathbb{S}^n$ with $\deg(f) = k$.

[1] This means: The index set I of our system of groups G_i is partially orderd and for each $i, j \in I$ there is a $k \in I$ with $i, j \leq k$.

Exercise 8.4 (Homology of products of spheres)

Recall that for well-pointed spaces X and Y there is a natural and split short exact sequence

$$0 \longrightarrow \tilde{H}_i(X) \oplus \tilde{H}_i(Y) \longrightarrow \tilde{H}_i(X \times Y) \longrightarrow \tilde{H}_i(X \wedge Y) \longrightarrow 0$$

for any $i \geq 0$.
a) Use this sequence to prove

$$\tilde{H}_i(\mathbb{S}^n \times X) \cong \begin{cases} \tilde{H}_i(X) & \text{for } 0 \leq i \leq n-1, \\ \mathbb{Z} \oplus \tilde{H}_n(X) & \text{for } i = n, \\ \tilde{H}_i(X) \oplus \tilde{H}_{i-n}(X) & \text{for } i \geq n+1. \end{cases} \tag{3}$$

b) Calculate the homology groups of $\mathbb{S}^n \times \mathbb{S}^m$. (Note: all groups are free and thus one needs only to calculate their rank.)

Exercise 8.5 (Homology of complex and quaternionic projective spaces)
Let $\mathbb{C}\mathrm{P}^n$ resp. $\mathbb{H}\mathrm{P}^n$ denote the complex resp. quaternionic projective spaces. Using the facts $\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^{n-1} \cong \mathbb{S}^{2n}$ resp. $\mathbb{H}\mathrm{P}^n/\mathbb{H}\mathrm{P}^{n-1} \cong \mathbb{S}^{4n}$ compute their homology groups:

$$H_k(\mathbb{C}\mathrm{P}^n) = \begin{cases} \mathbb{Z} & \text{for} \quad 0 \leq k \leq 2n, \ k \text{ even}, \\ 0 & \text{for} \quad 0 < k < 2n, \ k \text{ odd}, \\ 0 & \text{for} \quad k > 2n. \end{cases} \tag{4}$$

$$H_k(\mathbb{H}\mathrm{P}^n) = \begin{cases} \mathbb{Z} & \text{for} \quad 0 \leq k \leq 4n, \ k \equiv 0 \mod 4, \\ 0 & \text{for} \quad 0 < k < 4n, \ k \not\equiv 0 \mod 4, \\ 0 & \text{for} \quad k > 4n. \end{cases} \tag{5}$$

Exercise 8.6* (Connected sums of manifolds)
Let M and N be two connected oriented manifolds of dimension d. Their connected sum $M\#N$ is defined as follows. Take embeddings $\alpha\colon \mathbb{D} \to M$ and $\beta\colon \mathbb{D} \to N$ of closed d-dimensional disks $\mathbb{D} = \mathbb{D}^d$ in each of them, where α preserves the orientation and β reverses the orientation, remove the image of the interiors $M' = M - \alpha(\mathrm{int}(\mathbb{D}))$, $N' = N - \beta(\mathrm{int}(\mathbb{D}))$ and identify along the boundary:

$$M\#N := (M' \sqcup N') \,/\, \sim$$

where we identify $\alpha(t) \in M$ with $\beta(t) \in N$ for all $t \in \partial\mathbb{D} = \mathbb{S}^{d-1}$. (Using the fact, that in a connected oriented manifold any two orientation-preserving embeddings of a disk are isotopic and likewise any two orientation-reserving embeddings are isotopic, one can show that the construction above is independent of the choices of α and β.) Obviously, certain properties of M and N carry over to $M\#N$: (path) connectedness, compactness, closedness (empty boundary), smoothness (not so easy), orientability.

Some homeomorphisms are obvious:

$M\#\,\mathbb{S}^d \cong M$

$M\#N \cong N\#M$

$L\#(M\#N) \cong (L\#M)\#N$

An orientable surface of genus g is a connected sum of $g \geq 1$ tori: $F_g = \mathbb{T}\#\ldots\#\mathbb{T}$.

Compute the homology $H_*(M\#N)$ in terms of $H_*(M)$ and $H_*(N)$ with the help of the Mayer-Vietoris sequence using $M\#N = M' \cup N'$ and $S = M' \cap N' \cong \mathbb{S}^{d-1}$. To make things easier, assume that the inclusions $\iota_1\colon S \to M'$ and $\iota_2\colon S \to N'$ induce trivial maps in homology. This is actually not an unrealistic assumption; it is satisfied, if M and N are both compact, without boundary and orientable.
