Aufgaben zur Topologie I

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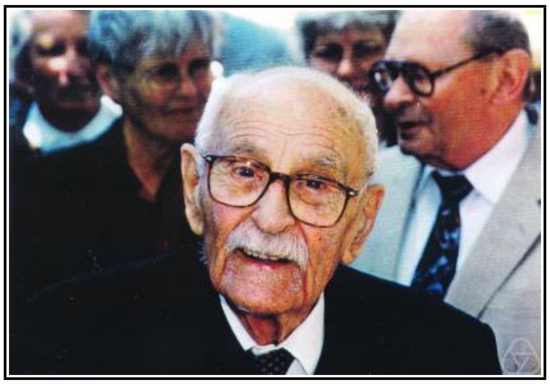
Blatt 7 due by: 27. 11. 2019



Leopold Vietoris (1891 - 2002), here at the age of 110 years. Together with Walther Mayer he found the Mayer-Vietoris sequence.

Exercise 7.1 (Relative Mayer-Vietoris sequence)
Let A and B be two open subspaces of X. Show that there is a natural long exact sequence

$$\ldots \longrightarrow H_k(X, A\cap B) \longrightarrow H_k(X,A)\oplus H_k(X,B) \longrightarrow H_k(X, A\cup B) \longrightarrow H_{k-1}(X, A\cap B) \longrightarrow \ldots$$

Exercise 7.2 (Homology of knot complements)
Let K be a knot in $\mathbb{R}^3$. Compute the homology groups of its complement.

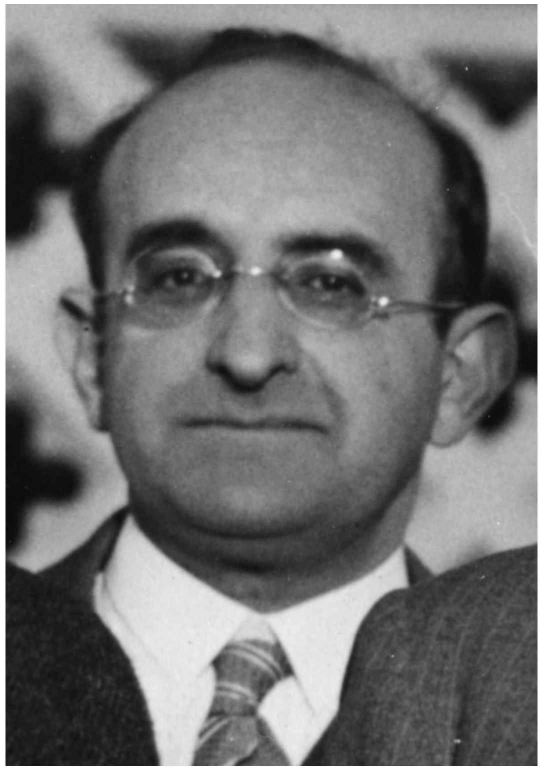
Exercise 7.3 (Homology of a colimit)
Let X_i be a sequence of spaces and let $f_i\colon X_i \to X_{i+1}$ be maps. There is a space $X = \lim X_i$, called the colimit (or direct limit) of these spaces. We write $g_i\colon X_i \to X$ for the canonical map. Describe this space and its universal property. Show that a subset $Y \subseteq X$ is closed if and only if its preimage $g_i^{-1}(Y)$ is a closed subset of X_i for every i.

Likewise in the category of abelian groups (or modules over $\mathbb{K}$) there is a colimit of a sequence of abelian groups A_i with homomorphisms $\phi_i\colon A_i \to A_{i+1}$. Describe this group and its universal property.

Now we assume that each of the maps f_i above is a closed embedding, (i.e., a homeomorphism onto its image, which is a closed subset of X_{i+1}) and that all the X_i are Hausdorff spaces.

- Show: If $K \subseteq X$ is a quasi-compact subspace, then there exists some $i \in \mathbb{N}$ such that K is contained in the image of $g_i\colon X_i \to X$. (Hint: Assume otherwise and construct an infinite subset $\{x_1, x_2, \ldots\}$ of K whose intersection with each image of g_i is finite, to derive a contradiction)
- Show: $S_n(\lim X_i) \cong \lim S_n(X_i)$
- Show: $H_n(\lim X_i) \cong \lim H_n(X_i)$ for all n.

- Show: $H_n(\mathbb{S}^\infty) = 0$ for all $n > 0$.
- If $D = \{x_1, x_2, \ldots\}$ is a discrete subset of $\mathbb{R}^m$, compute $H_n(\mathbb{R}^m - D)$ for all n.



Walther Mayer (1887 - 1948), Austrian mathematician and a pioneer of homology theory. He was for a long time a collaborator of Einstein.

Exercise 7.4 (Product, Wedge, and smash product of two spaces)
We call a pair (X, x_0) well-based, if there is a neighbourhood of the basepoint x_0 which is contractible on to x_0. Show: if (X, x_0) and (Y, y_0) are well-based, so is their product $(X \times Y, (x_0, y_0))$ and their wedge $X \vee Y$. We now assume that $X \vee Y \subseteq X \times Y$ is also a neighborhood deformation retract (a sufficient condition for that is that (X, x_0) and (Y, y_0) are 'strong NDR pairs', see Exercise 7.7 below).
Apply this to derive a natural short exact sequence

$$0 \longrightarrow H_n(X) \oplus H_n(Y) \longrightarrow H_n(X \times Y) \longrightarrow H_n(X \wedge Y) \longrightarrow 0$$

for $n \geq 1$. Furthermore, this sequence is split, even naturally split.
(Hint: Although there is in general no retraction $X \times Y \to X \vee Y$, there is a retraction of abelian groups $H_n(X \times Y) \to H_n(X) \oplus H_n(Y)$.)
Example: $X = Y = \mathbb{S}^1$. Compute H_2 of a torus.

Exercise 7.5 (Contractibility of the chain complex of degenerate simplices)
Let X be a topological space and $C(X)$ its associated singular chain complex. On exercise sheet 5 we defined the subcomplex $D(X)$ of $C(X)$. The goal of this exercise is to show that it is contractible. For this we first define a new chain complex $N(X)$, given in degree n by

$$N_n(X) = \bigcap_{i=0}^{n-1} \ker(d^i) \subseteq C_n(X),$$

where $d^i \colon C_n(X) \to C_{n-1}(X)$ is the i-th face operator. Show the following:

(1) The restriction of the differential ∂ to $N_n(X)$ is given by $(-1)^n d^n$.

(2) The $N_n(X)$ do indeed form a subcomplex of $C(X)$.

Next we define $F^p(X)_n$, for $p \geq 0$, as the subgroup of $C_n(X)$ given by

$$F^p(X)_n = \{x \in C_n(X) \mid d^i(x) = 0 \text{ for } 0 \leq i < \min(n,p)\}$$

and define maps $f^p : F^p(X)_n \to F^{p+1}(X)_n$ via

$$x \mapsto \begin{cases} x, & \text{if } n \leq p \\ x - s^p d^p(x) & \text{if } n > p. \end{cases}$$

Show the following:

(3) The $F^p(X)$ are subcomplexes of $C(X)$ and $N(X) = \bigcap_{p \geq 0} F^p(X)$.

(4) The maps f^p are chain homotopy inverses of the inclusion maps $i^p \colon F^{p+1}(X) \to F^p(X)$.

We can now define $f \colon C(X) \to N(X)$ by letting $f_n \colon C_n(X) \to N_n(X)$ be the composite $f^{n-1} \circ f^{n-2} \circ \ldots \circ f^0$.

(5) Show that f is well-defined and a chain homotopy-equivalence.

(6) The kernel of f is given by $D(X)$.

(7) Deduce that there is a natural splitting $C(X) \cong N(X) \oplus D(X)$ and that $D(X)$ is contractible.

Note also that this shows that the normalized chain complex $C(X)/D(X)$ is isomorphic to $N(X)$.

Exercise 7.6* (Complement of two circles in $\mathbb{R}^3$)
Assume A and B are two disjoint circles in $\mathbb{R}^3$. Compute the homology of the complement $X := \mathbb{R}^3 - (A \cup B)$ in case (1) A and B are unlinked, and in case (2) A and B are linked.

Exercise 7.7* (Strong NDR-pairs)
We say that a pair (X, A) is a *strong NDR-pair*, if there is a continuous function $u \colon X \to [0,1]$ and a homotopy $H \colon X \times [0,1] \to X$ such that the following hold:

- $A = u^{-1}(0)$.
- $H(x,0) = x$ for all $x \in X$.
- $H(a,t) = a$ for all $a \in A$ and $t \in [0,1]$.
- $H(x,1) \in A$ for all $x \in X$ satisfying $u(x) < 1$.
- If $u(x) < 1$, then $u(H(x,t)) < 1$ for all $t \in [0,1]$.

Show:

(1) If (X, A) is a strong NDR-pair, then A is a neighborhood deformation retract in X.

(2) If (X, A) and (Y, B) are strong NDR-pairs, then so is $(X \times Y, (A \times Y) \cup (X \times B))$.

Maths party: The student council of mathematics will organize the Maths Party on 28/11 in N8schicht. The presale will be held on Tue 26/11, Wed 27/11 and Thu 28/11 in the mensa Poppelsdorf. Further information can be found on fsmath.uni-bonn.de.