Aufgaben zur Topologie I

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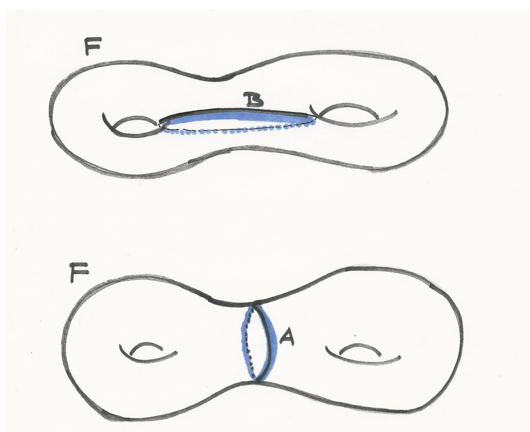
Blatt 6 due by: 20.11.2019

Exercise 6.1 (A surface and a curve)
Let F be a surface of genus 2 and let A and B the subspaces as indicated in the two drawings. Both subspaces are curves, so homoemorphic to $\mathbb{S}^1$; but A is separating and B is non-separating. Consider the long exact sequence of the pair (F, A) resp. of (F, B) and compute as many homology groups as possible. (Note: In the lecture we did this already for the subspace A; use your notes as a warm-up.)



A surface of genus 2 and two curves A and B.

Exercise 6.2 (Elementary Divisor Theorem for abelian groups)
The **Elementary Divisor Theorem** (or Smith Normal Form) says: *Any finitely generated abelian group C is isomorphic to a direct sum of cyclic groups.*

The summands are isomorphic to $\mathbb{Z}$ or to $\mathbb{Z}/n$ for some $n > 1$. By the Chinese Remainder Theorem, any $\mathbb{Z}/n$ is isomorphic to

$$\mathbb{Z}/n \cong \mathbb{Z}/{p_1}^{k_1} \oplus \ldots \oplus \mathbb{Z}/{p_r}^{k_r}, \quad \text{if } n = {p_1}^{k_1} \cdot \ldots \cdot {p_r}^{k_r}$$

is the prime decomposition of n. Therefore A decomposes into a direct sum of infinite cyclic modules and finite modules of prime power order. If we denote the first number by $R_0(C)$ and the second numbers by $R_{p^k}(C)$, then theses numbers form a complete set of invariants.
You can find the proof in many books, i.e., in Hilton-Wylie, pp. 158 - 160. (The theorem is true for finitely generated modules over any principal domain.) The proof uses the Stacked Bases Theorem: *If A is a submodule of a free module B of rank n, then A is also free of rank $m \leq n$ and there is a basis $a_1, \ldots, a_m$ of A and a basis $b_1, \ldots, b_n$ of B and positive numbers $\lambda_1, \ldots, \lambda_m$, such that*
(1) $a_i = \lambda_i b_i$ for $i = 1, \ldots, m$, and
(2) λ_i divides λ_{i+1} for $i = 1, \ldots, m-1$.

5.1.14 *Example.* The abelian group G is generated by g_1, g_2, g_3 subject to the relations $2g_1+2g_2+3g_3=0$, $3g_1-6g_3=0$. Let $F=(\gamma_1,\gamma_2,\gamma_3)$, $R=(\rho_1,\rho_2)$, where $\rho_1=2\gamma_1+2\gamma_2+3\gamma_3$, $\rho_2=3\gamma_1-6\gamma_3$; then $G\cong F/R$.

We transform the relation matrix $\mathbf{P}$ to diagonal form and make the corresponding changes in the bases for F, R explicitly. We thus find the rank and invariant factors of G and at the same time a set of generators for G arising from the decomposition 5.1.4.

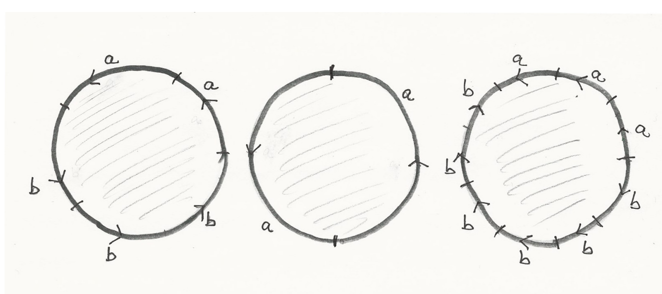
$$
\begin{array}{lll}
\begin{pmatrix} 2 & 2 & 3 \\ 3 & 0 & -6 \end{pmatrix} & (\gamma_1,\gamma_2,\gamma_3) & (\rho_1,\rho_2) \\
\downarrow & \downarrow & \downarrow \\
\begin{pmatrix} 2 & 2 & 1 \\ 3 & 0 & -6 \end{pmatrix} & (\gamma_1,\gamma_2+\gamma_3,\gamma_3) & (\rho_1,\rho_2) \\
\downarrow & \downarrow & \downarrow \\
\begin{pmatrix} 1 & 2 & 2 \\ -6 & 3 & 0 \end{pmatrix} & (\gamma_3,\gamma_1,\gamma_2+\gamma_3) & (\rho_1,\rho_2) \\
\downarrow & \downarrow & \downarrow \\
\begin{pmatrix} 1 & 0 & 0 \\ -6 & 15 & 12 \end{pmatrix} & (2\gamma_1+2\gamma_2+3\gamma_3,\gamma_1,\gamma_2+\gamma_3) & (\rho_1,\rho_2) \\
\downarrow & \downarrow & \downarrow \\
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 15 & 12 \end{pmatrix} & (2\gamma_1+2\gamma_2+3\gamma_3,\gamma_1,\gamma_2+\gamma_3) & (\rho_1,\rho_2+6\rho_1) \\
\downarrow & \downarrow & \downarrow \\
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 12 \end{pmatrix} & (2\gamma_1+2\gamma_2+3\gamma_3,\gamma_1,\gamma_1+\gamma_2+\gamma_3) & (\rho_1,\rho_2+6\rho_1) \\
\downarrow & \downarrow & \downarrow \\
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} & (2\gamma_1+2\gamma_2+3\gamma_3,5\gamma_1+4\gamma_2+4\gamma_3,\gamma_1+\gamma_2+\gamma_3) & (\rho_1,\rho_2+6\rho_1)
\end{array}
$$

The first step produced an equivalent matrix containing as an element the greatest common divisor of the elements of the original matrix; in general, this process may involve several steps. The second step placed this element in the leading position; the next two steps cleared the rest of the first row and column. The fifth step produced an equivalent matrix containing as an element the greatest common divisor of the elements other than the leading element; the sixth step cleared the rest of the second row.

It follows that $G=J\oplus J_3$, and that we may take J to be generated by $g_1+g_2+g_3$ and J_3 to be generated by $5g_1+4g_2+4g_3$. The reader is reminded that the generators are not unique nor even is the subgroup J.

An exampe of the Elementary Divisor Theorem, from Hilton-Wylie: *Homology Theory*, p. 163.
Peter Hilton (1923 - 2010) was a british mathematician. As a young man he worked during World War II in the british decoding department in Bletchley park; he occurs in the movie *The Imitation Game* on Alan Turing. Hilton is famous for his contributions to homotopy theory. Shaun Wylie, although less famous, was the PhD supervisor of Frank Adams.

a) Learn the above.
b) Read the example.
c) Then compute the simplicial homology $H_1^{\triangle}(X)$ of the space X we obtain by attaching three 2-cells to a figure-eight $\mathbb{S}^1 \vee \mathbb{S}^1$ (with one sphere denoted a, the other b) in the way the following drawing determines:



A space obtained by gluing a 5-gon, a 2-gon and 9-gon together.

Exercise 6.3 (Long exact sequence of a triple)
Let $X \supset Y \supset Z$ be a triple of spaces. Find a short exact sequence of chain complexes such that its long exact homology sequence is

$$\ldots \longrightarrow H_{n+1}(X,Y) \longrightarrow H_n(Y,Z) \longrightarrow H_n(X,Z) \longrightarrow H_n(X,Y) \longrightarrow H_{n-1}(Y,Z) \longrightarrow \ldots$$

The last (and the first) arrow is called the connecting homomorphism of the triple. Show that it factors as $H_n(X,Y) \to H_{n-1}(Y) \to H_{n-1}(Y,Z)$, where the first map is the connecting homomorphism of the pair (X,Y) and the second map is induced by the inclusion $(Y,\emptyset) \to (Y,Z)$.

Exercise 6.4 (Testing a cycle)
Assume A is a chain complex over $\mathbb{Z}$ and $a \in A_n$ is a cycle. We want to test, if a is a boundary or not. Consider the chain complex B obtained from A by attaching a new element b in degree $n+1$, to be precise: with $B_{n+1} = A_{n+1} \oplus \mathbb{Z}\langle b \rangle$ and $B_k = A_k$ for all $k \neq n+1$ and $\partial(b) = a$. We set $C = B/A$.

(1) Read off from the long exact homology sequence associated to $\mathcal{E}\colon 0 \to A \to B \to C \to 0$, if a is a boundary or not.

(2) Assume $H_n(B)$ and $H_n(A)$ are finitely generated and free. How can we conclude from the ranks of $H_n(A)$ and $H_n(B)$, whether a is a boundary ?

(3) Assume $H_n(B)$ is finitely generated. Show that $H_n(A)$ is finitely generated.

(4)* Still under the assumptions of (3), assume furthermore we know all the numbers R_0 and R_{p^k} for $H_*(A)$ and $H_*(B)$. Can we conclude, which multiples of a are boundaries ?

Exercise 6.5 (Tensor products of $\mathbb{K}$-modules)
Let A and B be two modules over a commutative ring $\mathbb{K}$ with unit. Their *tensor product* $A \otimes B$ is the $\mathbb{K}$-module defined as follows. We consider $A \times B$, the direct sum (or product); in the free $\mathbb{K}$-module $F_{\mathbb{K}}(A \times B)$ generated by this set we consider the relations:

- $(a_1 + a_2, b) \sim (a_1, b) + (a_2, b)$ for $a_1, a_2 \in A$ and $b \in B$;
- $(a, b_1 + b_2) \sim (a, b_1) + (a, b_2)$ for $a \in A$ and $b_1, b_1 \in B$;
- $\lambda(a, b) \sim (\lambda a, b) \sim (a, \lambda b)$ for $\lambda \in \mathbb{K}$, $a \in A$ and $b \in B$.

Then we define $A \otimes_{\mathbb{K}} B := F_{\mathbb{K}}(A \times B)/\sim$. The equivalence class of (a,b) is denoted by $a \otimes b$ and is called a *basic tensor*.

The following properties (1) - (4) are obvious; prove only (5) und (6):

(1) $A \otimes_{\mathbb{K}} (B \otimes_{\mathbb{K}} C) \cong (A \otimes_{\mathbb{K}} B) \otimes_{\mathbb{K}} C$

(2) $A \otimes_{\mathbb{K}} B \cong B \otimes_{\mathbb{K}} A$

(3) $A \otimes_{\mathbb{K}} (\bigoplus_i B_i) \cong \bigoplus_i (A \otimes_{\mathbb{K}} B_i)$

(4) $0 \otimes_{\mathbb{K}} B \cong 0 \cong A \otimes_{\mathbb{K}} 0$

(5) $A \otimes_{\mathbb{K}} \mathbb{K} \cong A$

(6) $\mathrm{Hom}_{\mathbb{K}}(A \otimes_{\mathbb{K}} B, C) \cong \mathrm{Hom}_{\mathbb{K}}(A, \mathrm{Hom}_{\mathbb{K}}(B, C))$

This last property (6) is the universal property of the tensor product: *For any $\mathbb{K}$-bi-linear function $f\colon A \times B \to C$ there is exactly one $\mathbb{K}$-linear map $F\colon A \otimes B \to C$ such that $F(a \otimes b) = f(a,b)$.*

For the case $\mathbb{K} = \mathbb{Z}$ and $n > 1$ prove the following:

(7) $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n \cong \mathbb{Z}/n$, — well, this is a special case of (5).

(8) $\mathbb{Z}/n \otimes_{\mathbb{Z}} \mathbb{Z}/m \cong \mathbb{Z}/k$, where $k = \gcd(n,m)$ is the greatest common divisor of n and m.

(9) $\mathbb{Z}/n \otimes_{\mathbb{Z}} \mathbb{Q} \cong 0$.

The properties (3), (7) and (8) allow us to compute $A \otimes B$ for any two finitely generated abelian groups.

For the case $\mathbb{K} = \mathbb{F}$ a field, prove the following:
If $\{a_1, \ldots, a_n\}$ is a basis for the vector space A, and $\{b_1, \ldots, b_m\}$ a basis for the vector space B, then the basic tensors $a_i \otimes b_j$ for all i and j forms a basis for $A \otimes_{\mathbb{F}} B$. Thus $\dim(A \otimes_{\mathbb{F}} B) = \dim A \cdot \dim B$.

Exercise 6.6* (Decomposition of chain complexes)
(1) Let

$$C\colon \quad 0 \leftarrow C_0 \leftarrow C_1 \leftarrow \cdots \leftarrow C_{N-1} \leftarrow C_N \leftarrow 0$$

be a bounded chain complex of finitely generated free abelian groups. Show that it splits as a direct sum of finitely many subcomplexes, each of which is of the form

$$0 \leftarrow \mathbb{Z} \leftarrow 0 \qquad \text{or} \qquad 0 \leftarrow \mathbb{Z} \xleftarrow{k} \mathbb{Z} \leftarrow 0$$

for some non-zero $k \in \mathbb{Z}$, up to shifts to the left and right.
(Hint: Use the Elementy Divisor Theorem (Smith normal form) for integer matrices. But do this carefully: start with $\partial\colon C_1 \to C_0$, then $\partial\colon C_2 \to C_1$ and so on up to $\partial\colon C_N \to C_{N-1}$.)
(2) Show that, if we had started with a bounded chain complex of finite-dimensional vector spaces over a field $\mathbb{F}$ instead, then it splits as a direct sum of finitely many subcomplexes of just two types, namely $0 \leftarrow \mathbb{F} \leftarrow 0$ and $0 \leftarrow \mathbb{F} \xleftarrow{\text{id}} \mathbb{F} \leftarrow 0$.

(3) Thus any bounded chain complex of finite-dimensional vector spaces is isomorphic to one with chain modules of the form $C_n = B_n \oplus A_n \oplus B_{n-1}$, where B_n denotes the boundaries of degree n, and where the boundary operator

$$\partial\colon C_n = B_n \oplus A_n \oplus B_{n-1} \quad \twoheadrightarrow \quad B_{n-1} \quad \hookrightarrow \quad B_{n-1} \oplus A_{n-1} \oplus B_{n-2} = C_{n-1}$$

is the projection of C_n onto B_{n-1} composed with the inclusion of B_{n-1} into C_{n-1}. It follows that the homology is $H_n(C) \cong A_n$ for all n.