Witold Hurewicz (1904 - 1956), polish-american mathematician, one of the pioniers of homotopy theory. He studied in Warsaw, Vienna, Amsterdam and became a professor at the MIT.

**Exercise 5.1** (Snake Lemma)
Consider the following commutative diagram of modules and homomorphisms over $\mathbb{K}$.

\[
\begin{array}{ccccccccc}
0 & \to & A & \xrightarrow{\iota} & B & \xrightarrow{\pi} & C & \to & 0 \\
\downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & & \\
0 & \to & A' & \xrightarrow{\iota'} & B' & \xrightarrow{\pi'} & C' & \to & 0
\end{array}
\]

We assume that the upper and lower sequence is exact.

(1) Show that there is a connecting homomorphism $d$: $\ker(\gamma) \to \coker(\alpha)$.

(2) Show that there is an exact sequence

\[
0 \to \ker(\alpha) \to \ker(\beta) \to \ker(\gamma) \xrightarrow{d} \coker(\alpha) \to \coker(\beta) \to \coker(\gamma) \to 0
\]

where all unnamed maps are induced by $\iota$ and $\pi$ resp. by $\iota'$ and $\pi'$. 
Exercise 5.2 (Normalized chains)
In the singular chain complex $S(X)$ of a space $X$ we call a singular simplex $c$ degenerate, if it is of the form $c = c' \circ s_i$ for a degeneracy map $s_i : \Delta^n \to \Delta^{n-1}$ and some $(n-1)$-chain $c'$.

(1) Show that the degenerate chains generate a subcomplex $D(X) \subset S(X)$.

(2) Show that the subcomplex is natural, i.e., for any continuous map $f : X \to Y$ the chain map $S(f)$ sends the subcomplex $D(X)$ to the subcomplex $D(Y)$.

We call the quotient complex $S(X)/D(X)$ the normalized chain complex $\bar{S}(X)$.

(3) We will see on the next exercise sheet that $D(X)$ is contractible. Use this and the long exact homology sequence to prove: there is an isomorphism $H_n(X) = H_n(S(X)) \to H_n(\bar{S}(X))$ for each $n \geq 0$.

Exercise 5.3 (Basic category theory)
Find and state the definitions of (1) a category, (2) of a functor, and (3) of a natural transformation between two functors. When is a morphism called isomorphism, when are two objects in a category called isomorphic? What is an automorphism (and the group of automorphisms) of an object in a category? What is a contravariant functor, what is a covariant functor? What is an equivalence of categories?

You can consult, for example, the following books:
S. MacLane: *Categories for the working mathematician*.\(^1\)
C. Weibel: *An introduction to homological algebra*, Appendix A.
P. J. Hilton - U. Stammbach: *A Course in Homological Algebra*, chap. II.

Saunders MacLane (1909-2005), US-american mathematician. He was a Ph.D. student in Göttingen and later became professor in Chicago; jointly with Samuel Eilenberg he is the founder of homological algebra and of category theory.

Examples: Spaces and continuous maps form a category TOP; based spaces and based maps form a category TOP\(_0\). Forgetting the basepoint and sending $(X, x_0)$ to $X$ is a functor (called forgetful functor).

Modules and homomorphisms over a ring $\mathbb{K}$ form a category $\text{MOD} - \mathbb{K}$; chain complexes over $\mathbb{K}$ together with chain maps form a category $\text{ChMOD} - \mathbb{K}$; they are a subcategory inside the category of all chain functions. Exact sequences of modules or of chain complexes can be considered a category.

All coverings $p : X \to B$ of a fixed spaces $B$ can be considered as a category $\text{COV}(B)$, the morphisms being all commutative triangles:

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{p} & & \downarrow{q} \\
B & \xrightarrow{} & \\
\end{array}
$$

Show the following:

1. The Hurewicz homomorphism $\pi_1(X, x_0) \to H_1(X)$ is a natural transformation between which functors from which category to which?

2. Sending a group $G$ to its abelianization $G^{ab}$ is a functor from the category $\text{GRP}$ of groups to the category $\text{AbGRP}$ of abelian groups. The Hurewicz isomorphism is a natural equivalence between which functors?

3. The connecting homomorphism $\partial_* : H_n(C) \to H_{n-1}(A)$ is a natural transformation between which functors from which category to which?

4. The automorphism group of an object $p : X \to B$ in the category $\text{COV}(B)$ is the group of deck transformations of $p$.

Exercise 5.4 (Cones and suspensions of chain complexes)

First, recall what the cone and the suspension of a space $X$ is. The quotient $CX := X \times [0,1]/X \times \{1\}$ is called the cone of $X$. The top layer $X \times \{1\}$ of the product becomes the tip point of the cone; call it $N$ like north pole. Obviously, $CX$ is always contractible; do you see the contraction?

The bottom layer of the product is $X$, via the embedding $\iota : x \mapsto (x,0)$. If we identify this bottom to a point $S$, like south pole, we get the suspension $\Sigma X := CX/X = (X \times [0,1])/X \times \{1\} \sim N, X \times \{0\} \sim S$. It contains $X$ as the equator $x \mapsto (x, \frac{1}{2})$.

Examples: $\mathbb{CS}^{n-1} \cong \mathbb{D}^n$ and $\mathbb{SS}^{n-1} \cong \mathbb{S}^n$

**Remark:** The sequence of spaces $X \to CX \to \Sigma X$ is called a cofibration sequence in homotopy theory; it is something like a 'short exact sequence of spaces' and thus one of the basic notions of homotopy theory.

1. Now let $A$ be a chain complex over some ring $\mathbb{K}$. Its cone $C(A)$ is constructed as follows: We set $C(A)_n := A_n \oplus A_{n-1}$ for all $n$ and as differential $\partial$ we take

$$
\partial = \begin{pmatrix}
\partial^A & (-1)^n \text{id} \\
0 & \partial^A
\end{pmatrix} : A_n \oplus A_{n-1} \to A_{n-1} \oplus A_{n-2}
$$

in matrix block form. So for $a \in A_n$ and $a' \in A_{n-1}$ we have $\partial(a, a') = (\partial(a) + (-1)^n a', \partial(a'))$. Show that this is a chain complex and find a contraction.

2. The suspension $\Sigma A$ of $A$ is the shifted chain complex, so $(\Sigma A)_n := A_{n-1}$ with the obvious shifted differential. Obviously, $H_n(\Sigma A) = H_{n-1}(A)$ for all $n$.

3. Show: There is an obvious inclusion of $A$ into $C(A)$ with $C(A)/A \cong \Sigma A$.

4. So there is a short exact sequence

$$
\mathcal{E} : 0 \to A \to C(A) \to \Sigma A \to 0
$$

of chain complexes. Conclude that the connecting homomorphisms in the long exact homology sequence of $\mathcal{E}$ are isomorphisms for all $n$. 

3
Exercise 5.5 (Naturality of the connecting homomorphism)
Formulate and prove the naturality of the connecting homomorphisms in the long exact homology sequence of a short exact sequence of chain complexes.

Exercise 5.6* (Cubical homology) Recall the definitions of Exercise 4.4. We call a cubic \( n \)-chain \( c : \mathbb{I}^n \to X \) degenerate, if \( c \) factors, for some \( i = 1, \ldots, n \), through the projection

\[ p_i : \mathbb{I}^n \to \mathbb{I}^{n-1}, p_i(t_1, \ldots, t_n) = (t_1, \ldots, t_{i-1}, \hat{t}_i, t_{i+1}, \ldots, t_n) \]
onto all variables except the \( i \)-th variable, so \( c = c' \circ p_i \) for some cubic \((n-1)\)-chain \( c' \).

a) Show: the degenerate cubic chains form a subcomplex \( D(X) \subset K(X) \).

We call the quotient complex \( K^\Box(X) := K(X)/D(X) \) the **cubical homology complex** and call its homology \( H_n^\Box(X) \) the **cubical homology** of \( X \).

b) Compute the cubical homology of a point. — Ah !!! See, how wonderful, it has the homology we expect !

c) How can we compare the singular with the cubical homology ? in other words, is there a transformation \( T: H_n(X) \to H_n^\Box(X) \) ? Use the functions \( \theta_n : \mathbb{I}^n \to \Delta^n, (t_1, \ldots, t_n) \mapsto (s_0, s_1, \ldots, s_n) \) with \( s_0 = 1 - t_1, \ s_1 = t_1(1 - t_2), \ldots, s_{n-1} = t_1t_2 \cdots t_{n-1}(1 - t_n), \ s_n = t_1t_2 \cdots t_{n-1}t_n \) to find a chain map \( \Theta : S(X) \to K^\Box(X) \), which induces a natural transformation \( T \).

*If you want to see, why \( T \) is an isomorphism for all \( X \), see F. Toenniessen: Topologie, p. 248-260.*

d)* The tensor product \( C := A \otimes B \) of two chain complexes over a ring \( \mathbb{K} \) is defined by \( C_n := \bigoplus_{i+j=n} A_i \otimes B_j \).

The differential on the summand \( A_i \otimes B_j \) of \( C_n \) is

\[ \partial_C(a \otimes b) = \partial_A(a) \otimes b + (-1)^i a \otimes \partial_B(b) \]

for an elementary tensor. (This is a kind of Leibniz rule.)

Show that this is indeed a chain complex, i.e., \( \partial \circ \partial = 0 \) holds.

e)** For any two spaces we define a product

\[ \times : K_p(X) \otimes K_q(Y) \to K_{p+q}(X \times Y), \]

by sending the elementary tensor \( a \otimes b \) with \( a : \mathbb{I}^p \to X \) and \( b : \mathbb{I}^q \to Y \) to the cartesian product \( a \times b : \mathbb{I}^p \times \mathbb{I}^q \to X \times Y \). Show that this is a chain map \( \times : K(X) \otimes K(Y) \to K(X \times Y) \). It does descend to a chain map of quotient complexes

\[ \times : K_p^\Box(X) \otimes K_q^\Box(Y) \to K_{p+q}^\Box(X \times Y), \]

and so it induces a product in cubical homology

\[ \times : H_p^\Box(X) \otimes H_q^\Box(Y) \to H_{p+q}^\Box(X \times Y). \]

Is this product associative ? Does it have a unit ? Is it commutative ?