Aufgaben zur Topologie I

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Blatt 5 due by: 13.11.2019

Witold Hurewicz (1904 - 1956), polish-american mathematician, one of the pioniers of homotopy theory. He studied in Warsaw, Vienna, Amsterdam and became a professor at the MIT.

Exercise 5.1 (Snake Lemma)
Consider the following commutative diagrame of modules and homomorphisms over $\mathbb{K}$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\iota} & B & \xrightarrow{\pi} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow[\iota']{} & B' & \xrightarrow[\pi']{} & C' & \longrightarrow & 0 \end{array}$$

We assume that the upper and lower sequence is exact.

(1) Show that there is a *connecting homomorphism* $d\colon \ker(\gamma) \longrightarrow \operatorname{coker}(\alpha)$.

(2) Show that there is an exact sequence

$$0 \to \ker(\alpha) \to \ker(\beta) \to \ker(\gamma) \xrightarrow{d} \operatorname{coker}(\alpha) \to \operatorname{coker}(\beta) \to \operatorname{coker}(\gamma) \to 0$$

where all unnamed maps are induced by ι and π resp. by ι' and π'.

Exercise 5.2 (Normalized chains)
In the singular chain complex $S(X)$ of a space X we call a singular simplex c *degenerate*, if it is of the form $c = c' \circ s_i$ for a degeneracy map $s_i \colon \Delta^n \to \Delta^{n-1}$ and some $(n-1)$-chain c'.

(1) Show that the degenerate chains generate a subcomplex $D(X) \subset S(X)$.

(2) Show that the subcomplex is *natural*, i.e., for any continuous map $f \colon X \to Y$ the chain map $S(f)$ sends the subcomplex $D(X)$ to the subcomplex $D(Y)$.

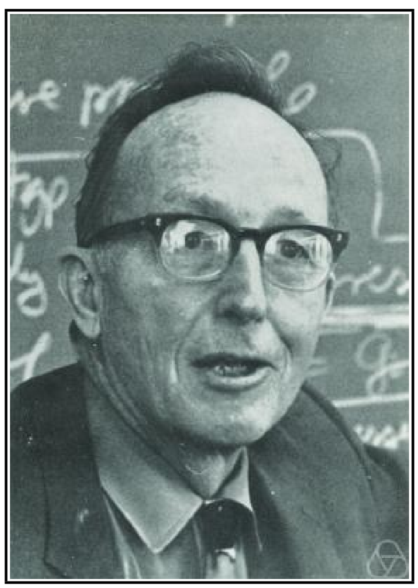
We call the quotient complex $S(X)/D(X)$ the *normalized chain complex* $\bar{S}(X)$.

(3) We will see on the next exercise sheet that $D(X)$ is contractible. Use this and the long exact homology sequence to prove: there is an isomorphism $H_n(X) = H_n(S(X)) \longrightarrow H_n(\bar{S}(X))$ for each $n \geq 0$.

Exercise 5.3 (Basic category theory)
Find and state the definitions of (1) a category, (2) of a functor, and (3) of a natural transformation betwen two functors. When is a morphism called isomorphism, when are two objects in a category called isomorphic ? What is an automorphism (and the group of automorphisms) of an object in a category ? What is a covariant functor, what is a contravariant functor ? What is an equivalence of categories ?

<u>You can consult, for example, the following books</u>:

S. MacLane: *Categories for the working mathematician.*[1]
A. Dold: *Lectures on Algebraic Topology*, Chap. I, §1.
A. Hatcher: *Algebraic Topology*, Chap. 2, inside Sect. 2.3. p. 162-165.
C. Weibel: *An introduction to homological algebra*, Appendix A.
P. J. Hilton - U. Stammbach: *A Course in Homological Algebra*, chap. II.
S. Lang: *Algebra*, chap. I §11.



Saunders MacLane (1909 - 2005), US-american mathematician. He was a Ph.D. student in Göttingen and later became professor in Chicago; jointly with Samual Eilenberg he is the founder of homological algebra and of category theory.

Examples: Spaces and continuous maps form a category TOP; based spaces and based maps form a category TOP_0. Forgetting the basepoint and sending (X, x_0) to X is a functor (called forgetful functor).

[1] Or — imitating a famous joke by J. F. Adams in *Infinite Loop Spaces*, p. 204 — consult J. MacNab: *Categories for the idle mathematician - all you need to know*, Proc. Philharmonic Soc. Zanzibar 17 (1976), pp. 10-9.

Modules and homomorphisms over a ring $\mathbb{K}$ form a category $\mathrm{MOD}-\mathbb{K}$; chain complexes over $\mathbb{K}$ together with chain maps form a category $\mathrm{ChMOD}-\mathbb{K}$; they are a subcategory inside the category of all chain functions.
Exact sequences of modules or of chain complexes can be considered a category.
All coverings $p : X \to B$ of a fixed spaces B can be considered as a category $\mathrm{COV}(B)$, the morphisms being all commutative triangles:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

Show the following:

(1) The Hurewicz homomorphism $\mathrm{hur}\colon \pi_1(X, x_0) \to H_1(X)$ is a natural transformation between which functors from which category to which ?

(2) Sending a group G to its abelianization G^{ab} is a functor from the category GRP of groups to the category AbGRP of abelian groups. The Hurewicz isomorphism is a natural equivalence between which functors ?

(3) The connecting homomorphism $\partial_*\colon H_n(C) \to H_{n-1}(A)$ is a natural transformation between which functors from which categrory to which ?

(4) The automorphism group of an object $p : X \to B$ in the category $\mathrm{COV}(B)$ is the group of deck transformations of p.

Exercise 5.4 (Cones and suspensions of chain complexes)
First, recall what the cone and the suspension of a space X is. The quotient $CX := X \times [0,1]/X \times \{1\}$ is called the *cone* of X. The top layer $X \times \{1\}$ of the product becomes the tip point of the cone; call it N like north pole. Obviously, CX is always contractible; do you see the contraction ?
The bottom layer of the product is X, via the embedding $\iota\colon x \mapsto (x, 0)$. If we identify this bottom to a point S, like south pole, we get the *suspension* $\Sigma X := CX/X = (X \times [0,1])/X \times \{1\} \sim N, X \times \{0\} \sim S$. It contains X as the equator $x \mapsto (x, \frac{1}{2})$.
Examples: $C\mathbb{S}^{n-1} \cong \mathbb{D}^n$ and $\Sigma\mathbb{S}^{n-1} \cong \mathbb{S}^n$

Remark: The sequence of spaces $X \to CX \to \Sigma X$ is called a *cofibration sequence* in homotopy theory; it is something like a 'short exact sequence of spaces' and thus one of the basic notions of homotopy theory.

(1) Now let A be a chain complex over some ring $\mathbb{K}$. Its *cone* $C(A)$ is constructed as follows: We set $C(A)_n := A_n \oplus A_{n-1}$ for all n and as differential ∂ we take

$$\partial = \begin{pmatrix} \partial^A & (-1)^n \,\mathrm{id} \\ 0 & \partial^A \end{pmatrix} : A_n \oplus A_{n-1} \longrightarrow A_{n-1} \oplus A_{n-2}$$

in matrix block form. So for $a \in A_n$ and $a' \in A_{n-1}$ we have $\partial(a, a') = (\partial(a) + (-1)^n a', \partial(a'))$. Show that this is a chain complex and find a contraction.

(2) The *suspension* ΣA of A is the shifted chain complex, so $(\Sigma A)_n := A_{n-1}$ with the obvious shifted differential. Obviously, $H_n(\Sigma A) = H_{n-1}(A)$ for all n.

(3) Show: There is an obvious inclusion of A into $C(A)$ with $C(A)/A \cong \Sigma A$.

(4) So there is a short exact sequence
$$\mathcal{E} : 0 \to A \to C(A) \to \Sigma A \to 0$$
of chain complexes. Conclude that the connecting homomorphisms in the long exact homology sequence of $\mathcal{E}$ are isomorphisms for all n.

Exercise 5.5 (Naturality of the connecting homomorphism)
Formulate and prove the naturality of the connecting homomorphisms in the long exact homology sequence of a short exact sequence of chain complexes.

Exercise 5.6* (Cubical homology) Recall the definitions of Exercise 4.4. We call a cubic n-chain $c\colon \mathbb{I}^n \to X$ *degenerate*, if c factors, for some $i = 1, \ldots, n$, through the projection

$$p_i\colon \mathbb{I}^n \to \mathbb{I}^{n-1}, p_i(t_1, \ldots, t_n) = (t_1, \ldots, t_{i-1}, \widehat{t_i}, t_{i+1}, \ldots, t_n)$$

onto all variables except the i-th variable, so $c = c' \circ p_i$ for some cubic $(n-1)$-chain c'.

a) Show: the degenerate cubic chains form a subcomplex $D(X) \subset K(X)$.

   We call the quotient complex $K^\square(X) := K(X)/D(X)$ the *cubical homology complex* and call its homology $H_n^\square(X)$ the *cubical homology* of X.

b) Compute the cubical homology of a point. — *Ah !!!! See, how wonderful, it has the homology we expect!*

c) How can we compare the singular with the cubical homology ? in other words, is there a transformation $T\colon H_n(X) \longrightarrow H_n^\square(X)$? Use the functions $\theta_n\colon \mathbb{I}^n \to \Delta^n, (t_1, \ldots, t_n) \mapsto (s_0, s_1, \ldots, s_n)$ with $s_0 = 1 - t_1,\ s_1 = t_1(1-t_2),\ \ldots,\ s_{n-1} = t_1 t_2 \cdots t_{n-1}(1-t_n),\ s_n = t_1 t_2 \cdots t_{n-1} t_n$ to find a chain map $\Theta\colon S(X) \to K^\square(X)$, which induces a natural transformation T.

   [*If you want to see, why T is an isomorphism for all X, see F. Toenniessen: Topologie, p. 248-260.*]

d)* The tensor product $C := A \otimes B$ of two chain complexes over a ring $\mathbb{K}$ is defined by $C_n := \bigoplus_{i+j=n} A_i \otimes B_j$. The differential on the summand $A_i \otimes B_j$ of C_n is

$$\partial^C(a \otimes b) = \partial^A(a) \otimes b + (-1)^j a \otimes \partial^B(b)$$

   for an elementary tensor. (This is a kind of Leibniz rule.)

   Show that this is indeed a chain complex, i.e., $\partial \circ \partial = 0$ holds.

e)** For any two spaces we define a product

$$\times\colon K_p(X) \otimes K_q(Y) \longrightarrow K_{p+q}(X \times Y),$$

   by sending the elementary tensor $a \otimes b$ with $a\colon \mathbb{I}^p \to X$ and $b\colon \mathbb{I}^q \to Y$ to the cartesian product $a \times b\colon \mathbb{I}^p \times \mathbb{I}^q \to X \times Y$. Show that this is a chain map $\times\colon K(X) \otimes K(Y) \to K(X \times Y)$. It does descend to a chain map of quotient complexes

$$\times\colon K_p^\square(X) \otimes K_q^\square(Y) \longrightarrow K_{p+q}^\square(X \times Y),$$

   and so it induces a product in cubical homology

$$\times\colon H_p^\square(X) \otimes H_q^\square(Y) \longrightarrow H_{p+q}^\square(X \times Y).$$

   Is this product associative ? Does it have a unit ? Is it commutative ?